Chapter 4
Risk-Averse Agent

What if the agent is risk-averse. Fluctuations of the agent's revenue stream occur because the principal's equipment unit can be either in state 0 ('operational') or in state 1 ('down'). In the operational state the penalty rate is 0, whereas in the down state the penalty rate is p. In other words, the penalty rate at any point of time can be modeled as pB where B is a Bernoulli random variable of value 0 with probability $P(0) = \mu/(\lambda+\mu)$ and value 1 with probability $P(1) = \lambda/(\lambda+\mu)$. The dispersion of B decreases as $P(1)$ moves away from $1/2$ in either direction. Denote momentarily $a \equiv P(1)$.

The risk of a random variable is often expressed by the dispersion of the underlying random fluctuation. Standard deviation is commonly used to measure the dispersion of revenue in risk sharing contracts because it is conveniently additive with the revenue stream (Stiglitz 1974; Fukunaga and Huffman 2009; Lewis and Bajari 2014). The standard deviation of pB as a function of a, is denoted by

$$s(a) \equiv \sigma_{pB} = p\sqrt{a(1-a)} \text{ for } a \in [0, 1]$$

We have modified the above risk measure somewhat. Since $s(a)$ strictly decreases as a moves away from $1/2$ in either direction so any other dispersion measure of pB that has this property is a monotone increasing function of the standard deviation $s(a)$. We choose to adopt the dispersion measure:

$$r(a) \equiv p\left(\frac{1}{2} - \left|\frac{1}{2} - a\right|\right) \text{ for } a \in [0, 1]$$

The $r(a)$ above is strictly decreasing as a gets away from $1/2$ in either direction and $r(a)$ has the property that for any $a, a' \in [0, 1]$, we have

$$r(a) \leq r(a') \Leftrightarrow s(a) \leq s(a')$$

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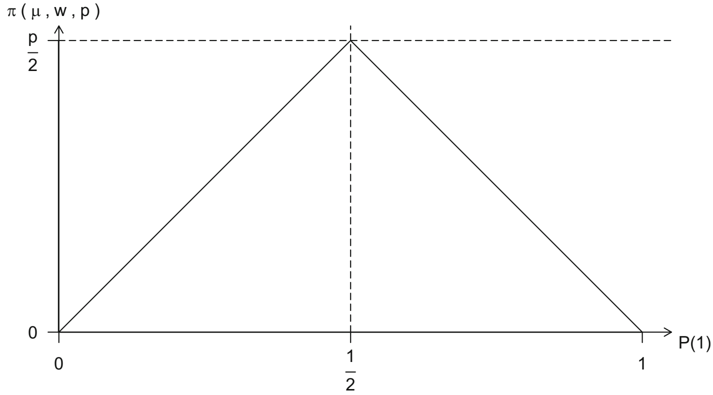


Fig. 4.1 $\pi(\mu, w, p)$ as a function of $P(1)$ when $\eta = 1$

Note that $r(a)$ increases (decreases) if and only if the standard deviation $s(a)$ increases (decreases).

Risk premium of a risk-averse agent is the \$ value he is willing to forfeit to avoid uncertainties (fluctuations) in his revenue stream and as a consequence the risk premium is defined as follows:

$$\pi(\mu, w, p) = \eta p \left(\frac{1}{2} - \left| \frac{1}{2} - a \right| \right) = \eta p \left(\frac{1}{2} - \left| \frac{1}{2} - P(1) \right| \right)$$
$$= \eta p \left(\frac{1}{2} - \left| \frac{1}{2} - \frac{\lambda}{\lambda + \mu} \right| \right) \tag{4.1}$$

Figure 4.1 is an example that depicts the shape of $\pi(\mu, w, p)$ as a function of $P(1)$ when $\eta = 1$. $\pi(\mu, w, p)$ reaches its peak when the equipment has equal likelihood of being operational and being failed. In such case the agent can hardly infer anything from the state of the equipment in order to predict his revenue stream and therefore it is considered the most risky. When the likelihood of the equipment being operational is close to 1, the agent can predict his revenue stream more precisely (less risky). Similarly when the likelihood of the equipment being failed is close to 1, the agent can also predict his revenue stream more precisely.

The real parameter η indicates the preference and intensity of the agent's risk attitude. When $\eta > 0$ the agent is risk-averse, when $\eta = 0$ the agent is risk-neutral (and the model reduces to the model of Chap. 3), and $\eta < 0$ indicates that the agent is risk-seeking (see Chap. 5). In the analysis below, the value η plays the role of an exogenous variable.

Modifying (3.2), the risk-averse agent's expected utility rate in this section is:

$$u_A(\mu; w, p) = \left(w - \frac{p\lambda}{\lambda + \mu} - \mu - \eta p \left(\frac{1}{2} - \left| \frac{1}{2} - \frac{\lambda}{\lambda + \mu} \right| \right) \right)_+$$
$$\text{for } w > 0, p > 0, \mu \geq 0 \tag{4.2}$$

Note that $\eta > 0 \Rightarrow \pi(\mu, w, p) \geq 0$, and such a risk premium being subtracted from a risk-neutral agent's expected utility rate (as in (4.2)) implies risk-aversion. The analysis is different for $\eta \in (0, 4/5)$ compared to $\eta \geq 4/5$. Thus, for convenience, when $\eta \in (0, 4/5)$ we describe the agent as *weakly risk-averse*, and when $\eta \geq 4/5$ we describe the agent as *strongly risk-averse*. We assume, say for historical reasons, that both the agent and the principal know not only the type of the risk-averse agent, but also the value of η.

The principal is always risk-neutral and her expression of expected profit rate $\Pi_P(w, p; \mu)$ is the same as (3.3).

Define the part inside the brackets in (4.2) as

$$u(\mu) \equiv w - \frac{p\lambda}{\lambda + \mu} - \mu - \eta p \left(\frac{1}{2} - \left| \frac{1}{2} - \frac{\lambda}{\lambda + \mu} \right| \right)$$
$$= \begin{cases} w - \eta p - \dfrac{(1 - \eta)p\lambda}{\lambda + \mu} - \mu, \ \mu \in [0, \lambda] \\ w - \dfrac{(1 + \eta)p\lambda}{\lambda + \mu} - \mu, \ \mu > \lambda \end{cases} \tag{4.3}$$

The behavior of the utility function $u(\mu)$ for $\mu \geq 0$ is of prime technical interest. Note that $u(\mu)$ is differentiable everywhere on $\mu \geq 0$ except at $\mu = \lambda$. When $\mu \in [0, \lambda)$:

$$\frac{du(\mu)}{d\mu} = \frac{(1 - \eta)p\lambda}{(\lambda + \mu)^2} - 1, \ \lim_{\mu \to 0+} \frac{du(\mu)}{d\mu} = \frac{1 - \eta}{\lambda} \left(p - \frac{\lambda}{1 - \eta} \right)$$
$$\lim_{\mu \to \lambda-} \frac{du(\mu)}{d\mu} = \frac{1 - \eta}{4\lambda} \left(p - \frac{4\lambda}{1 - \eta} \right) \text{ and } \frac{d^2u(\mu)}{d\mu^2} = -\frac{2(1 - \eta)p\lambda}{(\lambda + \mu)^3}$$

and when $\mu > \lambda$:

$$\frac{du(\mu)}{d\mu} = \frac{(1 + \eta)p\lambda}{(\lambda + \mu)^2} - 1, \ \lim_{\mu \to \lambda+} \frac{du(\mu)}{d\mu} = \frac{1 + \eta}{4\lambda} \left(p - \frac{4\lambda}{1 + \eta} \right)$$
$$\lim_{\mu \to +\infty} \frac{du(\mu)}{d\mu} = -1 \text{ and } \frac{d^2u(\mu)}{d\mu^2} = -\frac{2(1 + \eta)p\lambda}{(\lambda + \mu)^3} < 0$$

The above derivatives indicate the direction of monotonicity and the concavity/convexity of function $u(\mu)$ over $[0, \lambda)$ and $(\lambda, +\infty)$. Table 4.1 summarizes

Table 4.1 Indicators of the monotonicity and the concavity/convexity of function $u(\mu)$ in (4.3)

Case		Over $[0, \lambda)$			Over $(\lambda, +\infty)$		
		$u_\mu(0^+)$	$u(\mu)$ is	$u_\mu(\lambda^-)$	$u_\mu(\lambda^+)$	$u(\mu)$ is	$u_\mu(+\infty)$
$\eta \in \left(0, \frac{3}{5}\right]$	$p \in \left(0, \frac{\lambda}{1-\eta}\right]$	≤ 0	Concave	< 0	< 0	Concave	< 0
	$p \in \left(\frac{\lambda}{1-\eta}, \frac{4\lambda}{1+\eta}\right]$[a]	> 0	Concave	< 0	≤ 0	Concave	< 0
	$p \in \left(\frac{4\lambda}{1+\eta}, \frac{4\lambda}{1-\eta}\right]$	> 0	Concave	≤ 0	> 0	Concave	< 0
	$p \in \left(\frac{4\lambda}{1-\eta}, +\infty\right)$	> 0	Concave	> 0	> 0	Concave	< 0
$\eta \in \left(\frac{3}{5}, 1\right)$	$p \in \left(0, \frac{4\lambda}{1+\eta}\right]$	< 0	Concave	< 0	≤ 0	Concave	< 0
	$p \in \left(\frac{4\lambda}{1+\eta}, \frac{\lambda}{1-\eta}\right]$[b]	≤ 0	Concave	< 0	> 0	Concave	< 0
	$p \in \left(\frac{\lambda}{1-\eta}, \frac{4\lambda}{1-\eta}\right]$	> 0	Concave	≤ 0	> 0	Concave	< 0
	$p \in \left(\frac{4\lambda}{1-\eta}, +\infty\right)$	> 0	Concave	> 0	> 0	Concave	< 0
$\eta \in [1, +\infty)$	$p \in \left(0, \frac{4\lambda}{1+\eta}\right]$	< 0	Convex	< 0	≤ 0	Concave	< 0
	$p \in \left(\frac{4\lambda}{1+\eta}, +\infty\right)$	< 0	Convex	< 0	> 0	Concave	< 0

[a] Note that $\eta \in (0, 3/5] \Rightarrow 4\lambda/(1+\eta) \geq \lambda/(1-\eta)$
[b] Note that $\eta \in (3/5, 1) \Rightarrow \lambda/(1-\eta) > 4\lambda/(1+\eta)$

these indicators for various regions of the space $\mathbb{R}_+^2$ of the pairs (η, p). In the table $u_\mu(\cdot) = \lim_{\mu \to (\cdot)} du/d\mu$, and $u_\mu(\cdot^+)$ represents the limit of $u_\mu(\mu)$ as μ approaches $(\cdot)$ from above, and similar for $u_\mu(\cdot^-)$.

4.1 Optimal Strategies with a Weakly Risk-Averse Agent

Similarly to the risk-neutral agent case, agent's expected utility rate increases and principal's expected profit rate decreases in w, therefore for any value of p the principal maximizes her expected profit rate by lowering the compensation rate w yet maintaining the agent's participation by setting the agent's expected utility rate equal to his reservation utility rate. Although the principal cannot contract directly on the agent's service capacity, she anticipates the agent to optimize his expected utility rate when offered a contract. That is, for any w and p proposed by the principal, the agent computes his value of μ that maximizes his expected utility rate and subsequently decides whether to accept the contract or not, by solving the following optimization problem:

$$\max_{\mu \geq 0} u(\mu) = \max_{\mu \geq 0} \left\{ w - \frac{p\lambda}{\lambda + \mu} - \mu - \eta p \left(\frac{1}{2} - \left| \frac{1}{2} - \frac{\lambda}{\lambda + \mu} \right| \right) \right\} \tag{4.4}$$

The agent's optimal service capacity is denoted by $\mu^*(w, p) = \operatorname{argmax}_{\mu \geq 0} u(\mu)$.

Notation:

$$p_1 \equiv \frac{\lambda}{1 + \eta}, p_2 \equiv \frac{\lambda}{1 - \eta}, \text{ and } p_3 \equiv \frac{8\left(1 - \sqrt{1 - \eta^2}\right)\lambda}{\eta^2} \tag{4.5}$$

and the following identity is easily verified using the definition of p_3:

$$w_3 \equiv \eta p_3 + 2\sqrt{(1 - \eta) p_3 \lambda} - \lambda = 2\sqrt{(1 + \eta) p_3 \lambda} - \lambda \tag{4.6}$$

p_1, p_2, p_3 and w_3 are functions of λ and η. However we suppress the parameters (λ, η).

Next we state a number of technical lemmas (see proofs in the Appendix).

Lemma 4.1. *Let* $1 > \eta > 0$ *and* $\lambda > 0$. *If* $p \geq \lambda/(1 - \eta)$, *then* $p \geq \eta p + 2\sqrt{(1 - \eta)p\lambda} - \lambda > 0$.

Lemma 4.2. *Let* $1 > \eta > 0$ *and* $\lambda > 0$.

(a) *If* $p > 8\left(1 - \sqrt{1 - \eta^2}\right)\lambda/\eta^2$, *then* $\eta p - 2\left(\sqrt{1 + \eta} - \sqrt{1 - \eta}\right)\sqrt{p\lambda} > 0$.

(b) *If* $8\left(1 - \sqrt{1 - \eta^2}\right)\lambda/\eta^2 > p > 0$, *then* $0 > \eta p - 2\left(\sqrt{1 + \eta} - \sqrt{1 - \eta}\right)\sqrt{p\lambda}$.

(c) *If* $p = 8\left(1-\sqrt{1-\eta^2}\right)\lambda/\eta^2$, *then* $\eta p - 2\left(\sqrt{1+\eta}-\sqrt{1-\eta}\right)\sqrt{p\lambda} = 0$.

Lemma 4.3. *Let* $1 > \eta > 0$ *and* $\lambda > 0$, *then* $4\lambda/(1-\eta) > 8\left(1-\sqrt{1-\eta^2}\right)\lambda/\eta^2 > 4\lambda/(1+\eta)$.

Lemma 4.4. *Let* $\eta > 0$ *and* $\lambda > 0$. *If* $p > 4\lambda/(1+\eta)$, *then* $2\sqrt{(1+\eta)p\lambda} - \lambda > 0$.

Lemma 4.5. *Let* $\eta > 0$ *and* $\lambda > 0$.

(a) *If* $\left(1+2\eta+2\sqrt{\eta(1+\eta)}\right)\lambda > p > \left(1+2\eta-2\sqrt{\eta(1+\eta)}\right)\lambda$, *then* $0 > p - 2\sqrt{(1+\eta)p\lambda} + \lambda$.
(b) *If* $\left(1+2\eta-2\sqrt{\eta(1+\eta)}\right)\lambda > p > 0$ *or* $p > \left(1+2\eta+2\sqrt{\eta(1+\eta)}\right)\lambda$, *then* $p - 2\sqrt{(1+\eta)p\lambda} + \lambda > 0$.
(c) *If* $p = \left(1+2\eta-2\sqrt{\eta(1+\eta)}\right)\lambda$ *or* $\left(1+2\eta+2\sqrt{\eta(1+\eta)}\right)\lambda$, *then* $p - 2\sqrt{(1+\eta)p\lambda} + \lambda = 0$.

Lemma 4.6. *Let* $\eta > 0$ *and* $\lambda > 0$, *then* $4\lambda/(1+\eta) > \left(1+2\eta-2\sqrt{\eta(1+\eta)}\right)\lambda$.

Lemma 4.7. *Let* $\lambda > 0$.

(a) *If* $4/5 > \eta > 0$, *then* $\left(1+2\eta+2\sqrt{\eta(1+\eta)}\right)\lambda > \lambda/(1-\eta)$.
(b) *If* $1 > \eta > 4/5$, *then* $\lambda/(1-\eta) > \left(1+2\eta+2\sqrt{\eta(1+\eta)}\right)$.
(c) *If* $\eta = 4/5$, *then* $\left(1+2\eta+2\sqrt{\eta(1+\eta)}\right)\lambda = \lambda/(1-\eta)$.

Lemma 4.8. *Let* $\lambda > 0$.

(a) *If* $4/5 > \eta > 0$, *then* $8\left(1-\sqrt{1-\eta^2}\right)\lambda/\eta^2 > \lambda/(1-\eta)$.
(b) *If* $1 > \eta > 4/5$, *then* $\lambda/(1-\eta) > 8\left(1-\sqrt{1-\eta^2}\right)\lambda/\eta^2$.
(c) *If* $\eta = 4/5$, *then* $8\left(1-\sqrt{1-\eta^2}\right)\lambda/\eta^2 = \lambda/(1-\eta)$.

Lemma 4.8 part (a) implies $\eta \in (0, 4/5) \Rightarrow p_3 > p_2$, which makes condition (4.8) below consistent.

We identify the optimal response of a weakly risk-averse agent to any contract offer $(w, p) \in \mathbb{R}^2_+$ in Proposition 4.9.

Proposition 4.9. *Consider a weakly risk-averse agent* ($\eta \in (0, 4/5)$).

(a) *Given*

$$w \geq p \in (0, p_2] \tag{4.7}$$

then the agent accepts the contract and installs $\mu^*(w,p) = 0$ *resulting in expected utility rate* $u_A(\mu^*(w,p); w,p) = w - p \geq 0$. *The agent rejects the contract if* $p \in (0, p_2]$ *and* $w \in (0, p)$.

(b) *Given*

$$p \in (p_2, p_3) \text{ and } w \geq \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda \tag{4.8}$$

then the agent accepts the contract and installs $\mu^*(w,p) = \sqrt{(1-\eta)p\lambda} - \lambda > 0$ *resulting in expected utility rate* $u_A(\mu^*(w,p); w,p) = w - \eta p - 2\sqrt{(1-\eta)p\lambda} + \lambda \geq 0$*. The agent rejects the contract if* $p \in (p_2, p_3)$ *and* $w \in \left(0, \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda\right)$.

(c) *Given*

$$p = p_3 \text{ and } w \geq w_3 \tag{4.9}$$

then the agent accepts the contract and is indifferent about installing either $\mu^*(w,p) = \sqrt{(1-\eta)p_3\lambda} - \lambda$ *or* $\mu^*(w,p) = \sqrt{(1+\eta)p_3\lambda} - \lambda$*. In both cases the agent receives expected utility rate* $u_A(\mu^*(w,p); w,p) = w - w_3 \geq 0$*. If* $r \in (0, p_3)$*, then there exists a* w^* *such that* $\left((w^*, p_3), \sqrt{(1-\eta)p_3\lambda} - \lambda\right)$ *is the unique admissible solution (see Definition 2.3). If* $r = p_3$*, then there exists* w^* *such that* $\left((w^*, p_3), \sqrt{(1-\eta)p_3\lambda} - \lambda\right)$ *and* $\left((w^*, p_3), \sqrt{(1+\eta)p_3\lambda} - \lambda\right)$ *are both admissible solutions (see Definition 2.3). If* $r > p_3$*, then there exists a* w^* *such that* $\left((w^*, p_3), \sqrt{(1+\eta)p_3\lambda} - \lambda\right)$ *is the unique admissible solution (for proof see Proposition 4.12). He rejects the contract if* $p = p_3$ *and* $w \in (0, w_3)$.

(d) *Given*

$$p > p_3 \text{ and } w \geq 2\sqrt{(1+\eta)p\lambda} - \lambda \tag{4.10}$$

then the agent accepts the contract and installs $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda > 0$ *resulting in expected utility rate* $u_A(\mu^*(w,p); w,p) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda \geq 0$*. The agent rejects the contract if* $p > p_3$ *and* $w \in \left(0, 2\sqrt{(1+\eta)p\lambda} - \lambda\right)$.

Proof. According to Table 4.1, the optimization of $u(\mu)$ when $\eta \in (0, 3/5]$ versus $\eta \in (3/5, 4/5)$ is different. Therefore we prove the proposition separately for $\eta \in (0, 3/5]$ and $\eta \in (3/5, 4/5)$.

Case $\eta \in (0, 3/5]$: Note that $4p_2 > 4p_1 \geq p_2$ and according to Lemma 4.3, $4p_2 > p_3 > 4p_1$. Therefore we have $4p_2 > p_3 > 4p_1 \geq p_2$. Figure 4.2 shows the shape of $u(\mu)$ when $\eta \in (0, 3/5]$ and the value of p falls in different ranges. The structure of the proof when $\eta \in (0, 3/5]$ is depicted in Fig. 4.3.

Case $p \in (0, p_2]$: According to Table 4.1, $u(\mu)$ is decreasing with respect to $\mu \geq 0$. Thus the agent's optimal service capacity is $\mu^*(w,p) = 0$ and from (4.3) $u(\mu^*(w,p)) = w - p$.

Subcase $w \in (0, p)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.
Subcase $w \geq p$: $u(\mu^*(w,p)) \geq 0$, thus the agent would accept the contract if offered.

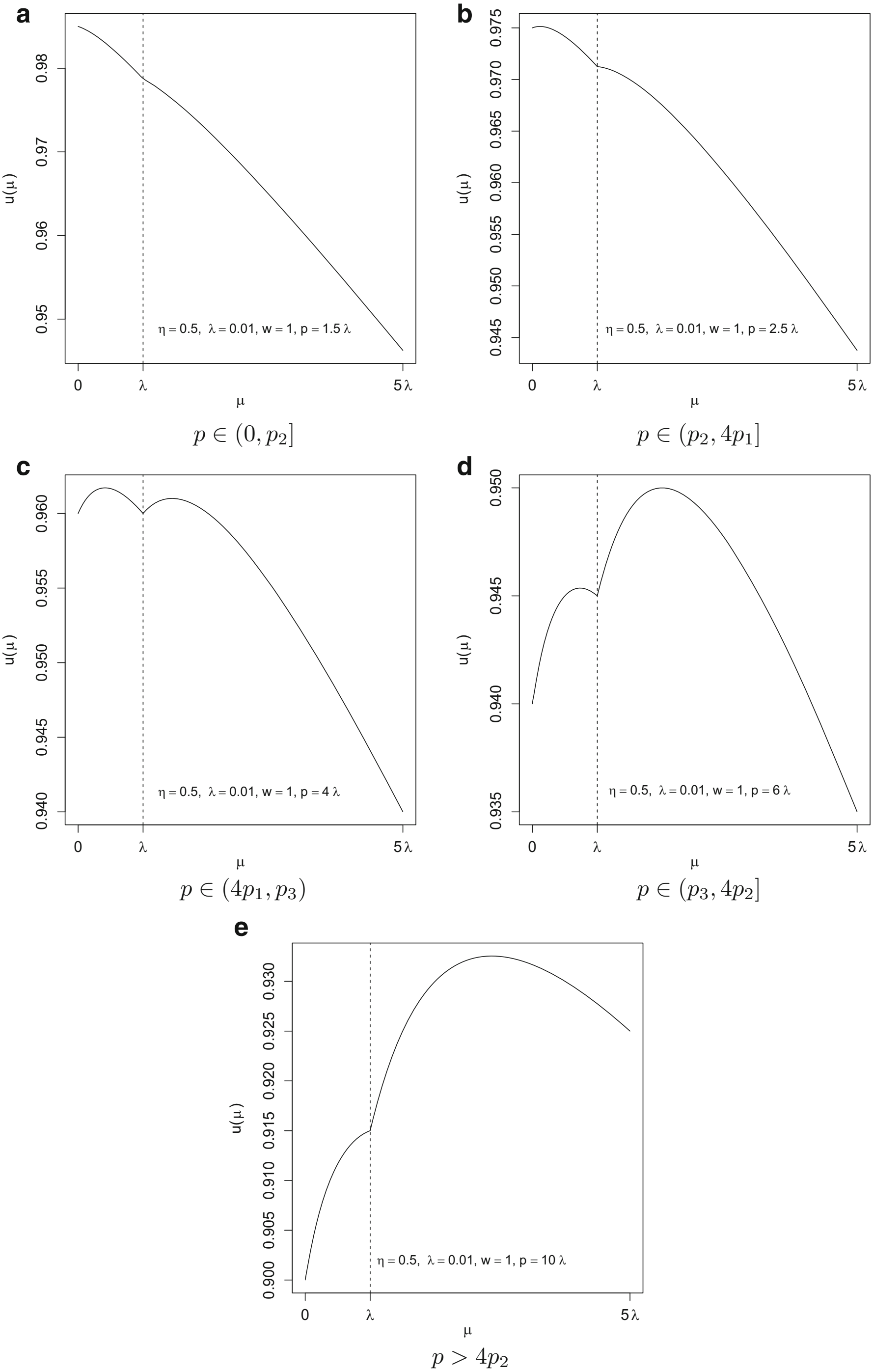


Fig. 4.2 Illustration of the forms of $u(\mu)$ when $\eta \in (0, 3/5]$

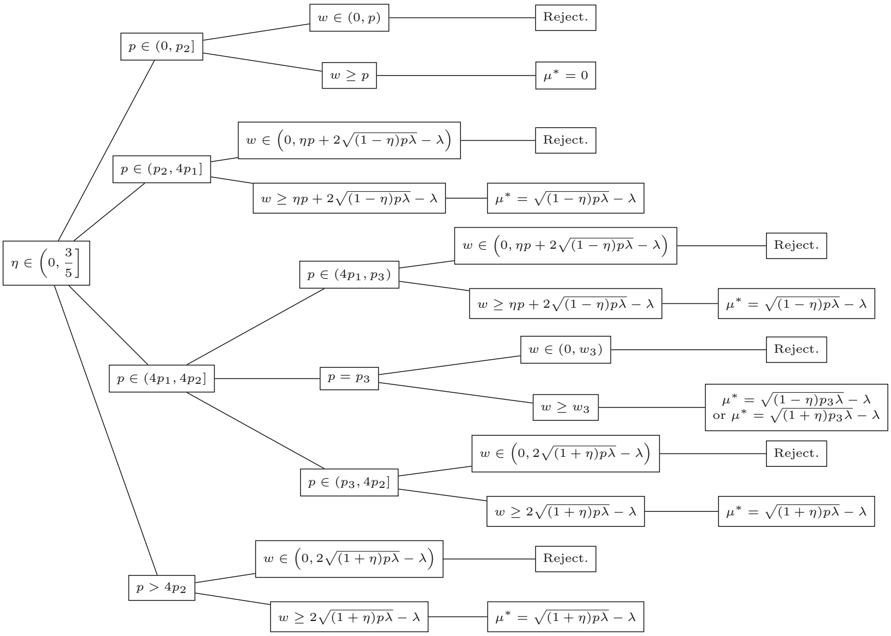


Fig. 4.3 Structure of the proof for Proposition 4.9 when $\eta \in (0, 3/5]$

Case $p \in (p_2, 4p_1]$: According to Table 4.1, the service capacity that maximizes $u(\mu)$ lies in $(0, \lambda)$. $\mu^*(w,p)$ is computed from first order condition $du(\mu)/d\mu|_{\mu=\mu^*(w,p)} = 0 \Rightarrow \mu^*(w,p) = \sqrt{(1-\eta)p\lambda} - \lambda > 0$ and from Eq. (4.3) $u(\mu^*(w,p)) = w - \eta p - 2\sqrt{(1-\eta)p\lambda} + \lambda$. According to Lemma 4.1, $p > p_2 \Rightarrow \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda > 0$.

Subcase $w \in \left(0, \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

Subcase $w \geq \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

Case $p \in (4p_1, 4p_2]$: According to Table 4.1, there is a service capacity that maximizes $u(\mu)$ for $\mu \in (0, \lambda]$ and a service capacity that maximizes $u(\mu)$ for $\mu > \lambda$. Denote the optimal service capacity in $(0, \lambda]$ by $\mu^*_{(0,\lambda]}(w,p)$. From the first order condition the optimal service capacity is $\mu^*_{(0,\lambda]}(w,p) = \sqrt{(1-\eta)p\lambda} - \lambda$ and from (4.3) $u\left(\mu^*_{(0,\lambda]}(w,p)\right) = w - \eta p - 2\sqrt{(1-\eta)p\lambda} + \lambda$. Denote the optimal service capacity for $\mu > \lambda$ by $\mu^*_\lambda(w,p)$, which is solved from first order condition $du(\mu)/d\mu|_{\mu=\mu^*_\lambda(w,p)} = 0 \Rightarrow \mu^*_\lambda(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and from (4.3) $u\left(\mu^*_\lambda(w,p)\right) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. The agent has a choice of two service capacities and he installs the one that generates a higher expected utility rate.

Note that $u\left(\mu_\lambda^*(w,p)\right) - u\left(\mu_{(0,\lambda]}^*(w,p)\right) = \eta p - 2\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)\sqrt{p\lambda}$. According to Lemma 4.3, $4p_2 > p_3 > 4p_1$, therefore we examine the following subcases.

Subcase $p \in (4p_1, p_3)$: From Lemma 4.2 part (b), $u\left(\mu_{(0,\lambda]}^*(w,p)\right) > u\left(\mu_\lambda^*(w,p)\right)$, thus the agent's optimal service capacity is $\mu^*(w,p) = \sqrt{(1-\eta)p\lambda} - \lambda$ and $u(\mu^*(w,p)) = w - \eta p - 2\sqrt{(1-\eta)p\lambda} + \lambda$. From Lemma 4.1, $p > 4p_1 \geq p_2 \Rightarrow \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda > 0$.

Subsubcase $w \in \left(0, \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

Subsubcase $w \geq \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, thus the agent would accept the contract if offered.

Subcase $p = p_3$: According to Lemma 4.2 part (c), $u\left(\mu_{(0,\lambda]}^*(w,p_3)\right) = u\left(\mu_\lambda^*(w,p_3)\right)$, indicating that installing $\mu_{(0,\lambda]}^*(w,p_3)$ or $\mu_\lambda^*(w,p_3)$ leads to the same agent's expected utility rate. Therefore the agent is indifferent about installing either $\mu^*(w,p) = \sqrt{(1-\eta)p_3\lambda} - \lambda$ or $\mu^*(w,p) = \sqrt{(1+\eta)p_3\lambda} - \lambda$. Still, the capacity value has to lead to admissible solutions (see Proposition 4.12). Recall the definition of w_3 from (4.6). According to Lemma 4.1, $p_3 > 4p_1 \geq p_2 \Rightarrow w_3 = \eta p_3 + 2\sqrt{(1-\eta)p_3\lambda} - \lambda > 0$.

Subsubcase $w \in (0, w_3)$: $u(\mu^*(w,p)) < 0$, thus the agent rejects the contract.

Subsubcase $w \geq w_3$: $u(\mu^*(w,p)) \geq 0$, thus the agent would accept the contract if offered.

Subcase $p \in (p_3, 4p_2]$: By Lemma 4.2 part (a), $u\left(\mu_\lambda^*(w,p)\right) > u\left(\mu_{(0,\lambda]}^*(w,p)\right)$, therefore the agent's optimal service capacity is $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and $u(\mu^*(w,p)) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to Lemma 4.4, $p > p_3 > 4p_1 \Rightarrow 2\sqrt{(1+\eta)p\lambda} - \lambda > 0$.

Subsubcase $w \in \left(0, 2\sqrt{(1+\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

Subsubcase $w \geq 2\sqrt{(1+\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

Case $p > 4p_2$: According to Table 4.1, the service capacity that maximizes $u(\mu)$ satisfies $\mu > \lambda$. From the first order condition the agent's optimal service capacity is $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and from Eq. (4.3) $u(\mu^*(w,p)) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to Lemma 4.4, $p > 4p_2 > 4p_1 \Rightarrow 2\sqrt{(1+\eta)p\lambda} - \lambda > 0$.

Subcase $w \in \left(0, 2\sqrt{(1+\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, thus the agent rejects the contract.

Subcase $w \geq 2\sqrt{(1+\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

This completes the proof for Proposition 4.9 when $\eta \in (0, 3/5]$.

Case $\eta \in (3/5, 4/5)$: Note that $4p_2 > p_2 > 4p_1$ and according to Lemmas 4.3 and 4.8 part (a), $4p_2 > p_3 > p_2$. Therefore we have $4p_2 > p_3 > p_2 > 4p_1$. Figure 4.4 shows the shape of $u(\mu)$ when $\eta \in (3/5, 4/5)$ and the value of p falls in different ranges. The structure of the proof when $\eta \in (3/5, 4/5)$ is depicted in Fig. 4.5.

Case $p \in (0, 4p_1]$: According to Table 4.1, $u(\mu)$ is decreasing with respect to $\mu \geq 0$. Thus the agent's optimal service capacity is $\mu^*(w, p) = 0$ and from (4.3) $u(\mu^*(w, p)) = w - p$.

Subcase $w \in (0, p)$: $u(\mu^*(w, p)) < 0$, therefore the agent rejects the contract.
Subcase $w \geq p$: $u(\mu^*(w, p)) \geq 0$, thus the agent would accept the contract if offered.

Case $p \in (4p_1, p_2]$: According to Table 4.1, there is a service capacity that maximizes $u(\mu)$ for $\mu \in [0, \lambda)$ and a service capacity that maximizes $u(\mu)$ for $\mu > \lambda$. Denote the optimal service capacity in $[0, \lambda)$ by $\mu^*_{[0,\lambda)}(w, p)$. Note that $u(\mu)$ is decreasing with respect to μ over $[0, \lambda)$, therefore the agent's optimal service capacity is $\mu^*_{[0,\lambda)}(w, p) = 0$ and from (4.3) $u\left(\mu^*_{[0,\lambda)}(w, p)\right) = w - p$. Denote the optimal service capacity for $\mu > \lambda$ by $\mu^*_{\lambda}(w, p)$. From the first order condition $\mu^*_{\lambda}(w, p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and from Eq. (4.3) $u\left(\mu^*_{\lambda}(w, p)\right) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. The agent has to choose one of the two service capacities and installs the one with higher expected utility rate. Note that $u\left(\mu^*_{\lambda}(w, p)\right) - u\left(\mu^*_{[0,\lambda)}(w, p)\right) = p - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to Lemma 4.6, $4p_1 > \left(1 + 2\eta - 2\sqrt{\eta(1+\eta)}\right)\lambda$ and according to Lemma 4.7 part (a), $\left(1 + 2\eta + 2\sqrt{\eta(1+\eta)}\right)\lambda > p_2$. Thus according to Lemma 4.5 part (a), $u\left(\mu^*_{[0,\lambda)}(w, p)\right) > u\left(\mu^*_{\lambda}(w, p)\right)$, the agent's optimal service capacity is $\mu^*(w, p) = 0$ and $u(\mu^*(w, p)) = w - p$.

Subcase $w \in (0, p)$: $u(\mu^*(w, p)) < 0$, therefore the agent rejects the contract.
Subcase $w \geq p$: $u(\mu^*(w, p)) \geq 0$, thus the agent would accept the contract if offered.

Case $p \in (p_2, 4p_2]$: According to Table 4.1, there is a service capacity that maximizes $u(\mu)$ for $\mu \in (0, \lambda]$ and a service capacity that maximizes $u(\mu)$ for $\mu > \lambda$. Denote the optimal service capacity in $(0, \lambda]$ by $\mu^*_{(0,\lambda]}(w, p)$. From the first order condition $\mu^*_{(0,\lambda]}(w, p) = \sqrt{(1-\eta)p\lambda} - \lambda$ and from Eq. (4.3) $u\left(\mu^*_{(0,\lambda]}(w, p)\right) = w - \eta p - 2\sqrt{(1-\eta)p\lambda} + \lambda$. Denote the optimal service capacity for $\mu > \lambda$ by $\mu^*_{\lambda}(w, p)$. From the first order condition $\mu^*_{\lambda}(w, p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and from Eq. (4.3) $u\left(\mu^*_{\lambda}(w, p)\right) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. The agent has to choose one of the two service capacities and installs the one that generates a higher expected utility rate. Note that $u\left(\mu^*_{\lambda}(w, p)\right) - u$

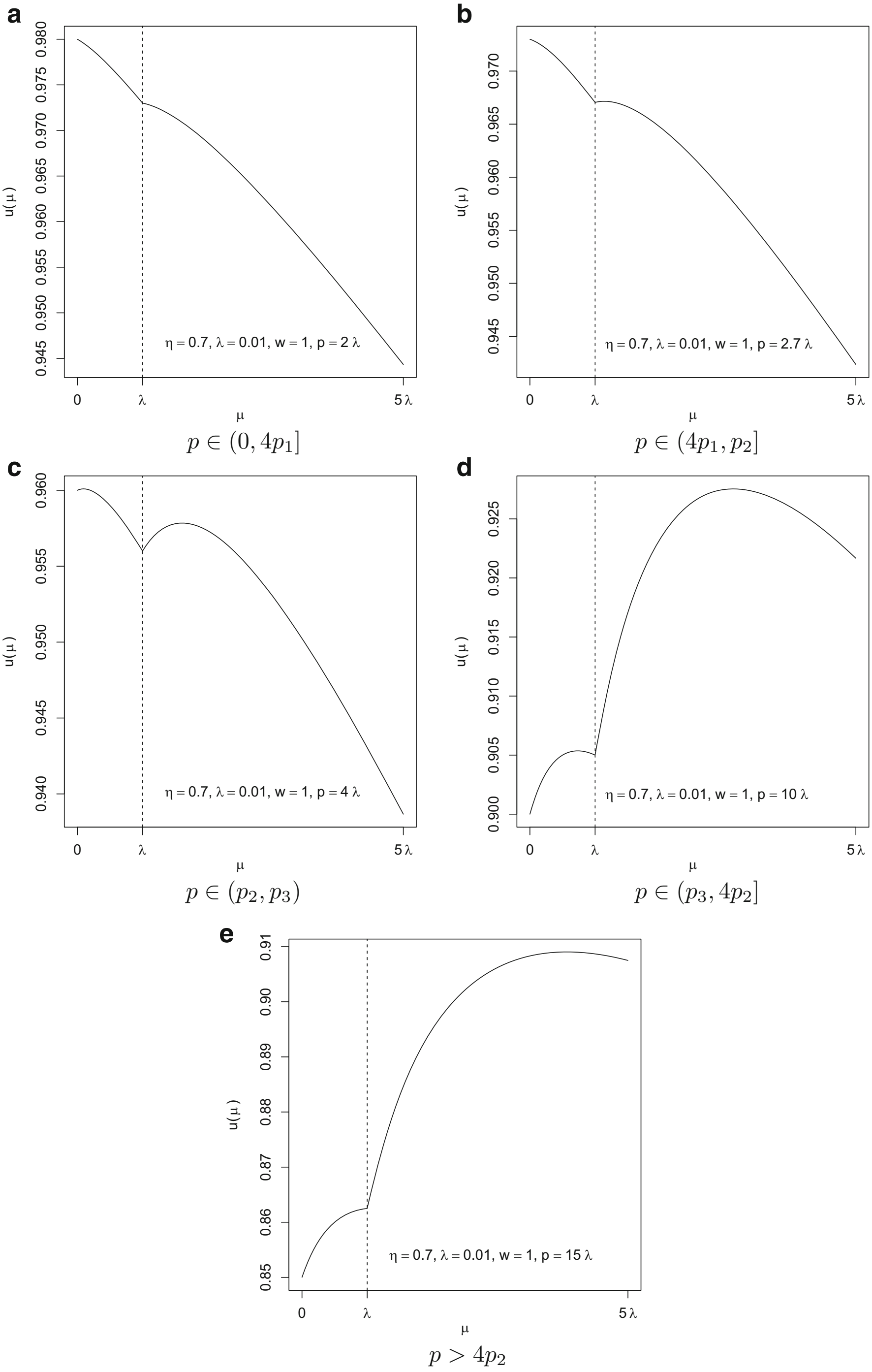


Fig. 4.4 Illustration of the forms of $u(\mu)$ when $\eta \in (3/5, 4/5)$

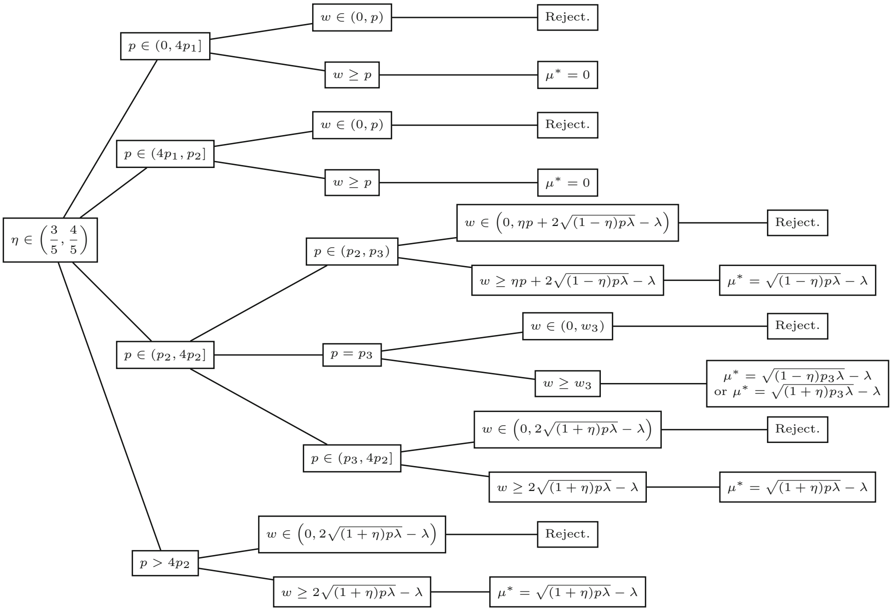


Fig. 4.5 Structure of the proof for Proposition 4.9 when $\eta \in (3/5, 4/5)$

$\left(\mu^*_{(0,\lambda]}(w,p)\right) = \eta p - 2\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)\sqrt{p\lambda}$. According to Lemmas 4.3 and 4.8 part (a), $4p_2 > p_3 > p_2$, therefore we examine the following subcases.

Subcase $p \in (p_2, p_3)$: By Lemma 4.2 part (b), $u\left(\mu^*_{(0,\lambda]}(w,p)\right) > u\left(\mu^*_\lambda(w,p)\right)$, therefore the agent's optimal service capacity is $\mu^*(w,p) = \sqrt{(1-\eta)p\lambda} - \lambda$ and $u(\mu^*(w,p)) = w - \eta p - 2\sqrt{(1-\eta)p\lambda} + \lambda$. According to Lemma 4.1, $p > p_2 \Rightarrow \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda > 0$.

Subsubcase $w \in \left(0, \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

Subsubcase $w \geq \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, thus the agent would accept the contract if offered.

Subcase $p = p_3$: According to Lemma 4.2 part (c), $u\left(\mu^*_{(0,\lambda]}(w,p_3)\right) = u\left(\mu^*_\lambda(w,p_3)\right)$, indicating that installing $\mu^*_{(0,\lambda]}(w,p)$ or $\mu^*_\lambda(w,p)$ leads to the same agent's expected utility rate. Therefore the agent is indifferent about installing $\mu^*(w,p) = \sqrt{(1-\eta)p_3\lambda} - \lambda$ or $\mu^*(w,p) = \sqrt{(1+\eta)p_3\lambda} - \lambda$. Again, the service capacity has to lead to admissible solutions (see Proposition 4.12). Recall the definition of w_3 from (4.6). According to Lemma 4.1, $p_3 > p_2 \Rightarrow w_3 = \eta p_3 + 2\sqrt{(1-\eta)p_3\lambda} - \lambda > 0$.

Subsubcase $w \in (0, w_3)$: $u(\mu^*(w,p)) < 0$, thus the agent rejects the contract.

Subsubcase $w \geq w_3$: $u(\mu^*(w,p)) \geq 0$, thus the agent would accept the contract if offered.

Subcase $p \in (p_3, 4p_2]$: By Lemma 4.2 part (a), $u\left(\mu^*_\lambda(w,p)\right) > u\left(\mu^*_{(0,\lambda]}(w,p)\right)$, thus the agent's optimal service capacity is $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and $u(\mu^*(w,p)) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to Lemma 4.4, $p > p_3 > p_2 > 4p_1 \Rightarrow 2\sqrt{(1+\eta)p\lambda} - \lambda > 0$.

Subsubcase $w \in \left(0, 2\sqrt{(1+\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

Subsubcase $w \geq 2\sqrt{(1+\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

Case $p > 4p_2$: According to Table 4.1, the service capacity that maximizes $u(\mu)$ satisfies $\mu > \lambda$. From the first order condition the agent's optimal service capacity is $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and from Eq. (4.3) $u(\mu^*(w,p)) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to Lemma 4.4, $p > 4p_2 > 4p_1 \Rightarrow 2\sqrt{(1+\eta)p\lambda} - \lambda > 0$.

Subcase $w \in \left(0, 2\sqrt{(1+\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, thus the agent rejects the contract.

Subcase $w \geq 2\sqrt{(1+\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

This completes the proof for Proposition 4.9 when $\eta \in (3/5, 4/5)$. □

In summary, given exogenous market conditions such that a contract offer satisfying the reservation value constraints for both the principal and a weakly risk-averse agent exists (see Theorem 4.19 and Proposition 4.20 later), the agent determines his optimal capacity using one of two formulas:

$$\mu^*(w,p) = \sqrt{(1-\eta)p\lambda} - \lambda > 0 \text{ or } \mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda > 0$$

The conditions when a weakly risk-averse agent accepts the contract can be depicted by the shaded areas in Fig. 4.6, where $\eta = 0.6$. The three shaded areas with different grey scales represent conditions (4.7), (4.8) and (4.10) under which the agent accepts the contract but responds differently. The lower bound function of the shaded areas (denoted by $w_0(p)$) represents the set of offers with agent's zero expected utility rate. $w_0(p)$ is defined as follows:

$$w_0(p) = \begin{cases} p \text{ when } p \in (0, p_2] \\ \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda \text{ when } p \in (p_2, p_3] \\ 2\sqrt{(1+\eta)p\lambda} - \lambda \text{ when } p > p_3 \end{cases}$$

Note that since $\lim_{p\to p_2^-} w_0(p) = \lim_{p\to p_2^+} w_0(p) = p_2$ and $\lim_{p\to p_3^-} w_0(p) = \lim_{p\to p_3^+} = \eta p_3 + 2\sqrt{(1-\eta)p_3\lambda} - \lambda$, $w_0(p)$ is continuous everywhere over interval

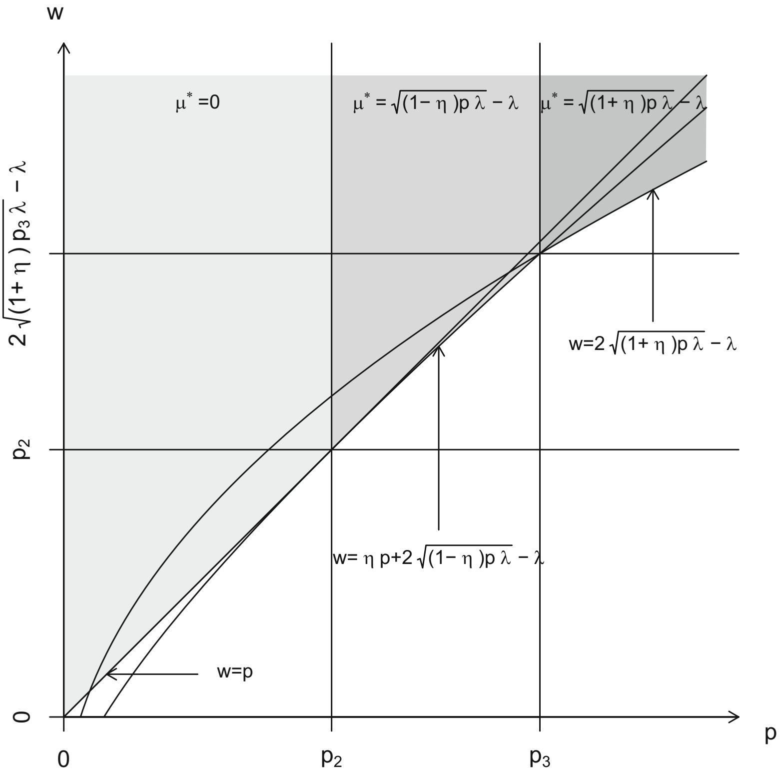


Fig. 4.6 Conditions when a weakly risk-averse agent accepts the contract with $\eta = 0.6$

$p \in \mathbb{R}_+$. Since $\lim_{p \to p_2^-} dw_0(p)/dp = \lim_{p \to p_2^+} dw_0(p)/dp = 1$, $w_0(p)$ is differentiable at $p = p_2$. However since $\lim_{p \to p_3^-} dw_0(p)/dp = \eta + \sqrt{(1-\eta)\lambda/p_3} \neq \sqrt{(1+\eta)\lambda/p_3} = \lim_{p \to p_3^+} dw_0(p)/dp$, $w_0(p)$ is not differentiable at $p = p_3$.

4.1.1 *Sensitivity Analysis of a Weakly Risk-Averse Agent's Optimal Strategy*

The principal would not propose an acceptable contract that results in $u_A(\mu^* = 0) \geq \underline{u_A} = 0$. Therefore the only viable cases to consider are when the agent accepts the contract and installs positive service capacities: $\mu^*(w, p) = \sqrt{(1-\eta)p\lambda} - \lambda$ or $\mu^*(w, p) = \sqrt{(1+\eta)p\lambda} - \lambda$. We examine the two viable contracts with positive service capacities.

First the case $\mu^*(w,p) = \sqrt{(1-\eta)p\lambda} - \lambda$. According to (4.8) the compensation rate w is bounded below by $\eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda = \eta pP(0) + pP(1) + \mu^*(w,p)$, with the term $\eta pP(0)$ representing the expected risk rate perceived by the agent and the term $pP(1)$ representing the expected penalty rate charged by the principal when the optimal capacity is installed. It indicates that the agent should at least be reimbursed for the expected risk rate, the expected penalty rate and the cost of the optimal service capacity.

The optimal service capacity $\sqrt{(1-\eta)p\lambda} - \lambda$ depends on p, λ, and η. Its derivatives are:

$$\frac{\partial\mu^*}{\partial p} = \sqrt{\frac{(1-\eta)\lambda}{4p}} > 0,\ \frac{\partial\mu^*}{\partial\lambda} = \sqrt{\frac{(1-\eta)p}{4\lambda}} - 1 \text{ and } \frac{\partial\mu^*}{\partial\eta} = -\sqrt{\frac{p\lambda}{4(1-\eta)}} < 0$$

These derivatives indicate that given a λ and η, the agent will increase his service capacity when the penalty rate increases. Note that $\sqrt{(1-\eta)p\lambda} - \lambda$, as a function of λ, decreases when $\lambda > (1-\eta)p/4$. From conditions (4.8) and (4.9) the agent installs service capacity $\sqrt{(1-\eta)p\lambda} - \lambda$ when $p \in (p_2, p_3]$ and from Lemma 4.3 we have $4p_2 > p_3$. Therefore we have $4\lambda/(1-\eta) = 4p_2 > p \Rightarrow \lambda > (1-\eta)p/4 \Rightarrow \partial\mu^*/\partial\lambda < 0$. Thus, given a p and η, the savings from reducing the service capacity are greater than the increase in the penalty charge and in the risk rate, and the agent will reduce μ when λ increases. Given a p and λ, the agent will reduce the μ when he is more risk-averse.

The agent's optimal expected utility rate when installing capacity $\mu^*(w,p) = \sqrt{(1-\eta)p\lambda} - \lambda$ is $u_A^* \equiv u_A(\mu^*(w,p); w,p) = w - \eta p - 2\sqrt{(1-\eta)p\lambda} + \lambda$, and it depends on w, p, η and λ. Note that $\partial u_A^*/\partial w = -1 < 0$, $\partial u_A^*/\partial p = -\eta - \sqrt{(1-\eta)\lambda/p} < 0$, indicating that the agent's optimal expected utility rate decreases with the compensation rate and the penalty rate. Note that $\partial u_A^*/\partial\eta = -\sqrt{p}\left(\sqrt{p} - \sqrt{p_2}\right)$ and $\partial u_A^*/\partial\lambda = -\left(\sqrt{p} - \sqrt{p_2}\right)/\sqrt{p_2}$, and from Proposition 4.9 $p > p_2 \Rightarrow \sqrt{p} - \sqrt{p_2} > 0$, therefore the agent's optimal expected utility rate also decreases with his risk intensity and the failure rate.

Next we examine the case $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$. According to (4.10) the compensation rate w is bounded below by $2\sqrt{(1+\eta)p\lambda} - \lambda = \eta pP(1) + pP(1) + \mu^*(w,p)$, with the term $\eta pP(1)$ representing the expected risk rate perceived by the agent and $pP(1)$ representing the expected penalty rate charged by the principal when the optimal capacity is installed. It indicates that the agent should at least be reimbursed for the expected risk rate, the expected penalty rate and the cost of the optimal service capacity.

The optimal service capacity $\sqrt{(1+\eta)p\lambda} - \lambda$ depends on p, λ, and η. Its derivatives are:

$$\frac{\partial\mu^*}{\partial p} = \sqrt{\frac{(1+\eta)\lambda}{4p}} > 0,\ \frac{\partial\mu^*}{\partial\lambda} = \sqrt{\frac{(1+\eta)p}{4\lambda}} - 1 \text{ and } \frac{\partial\mu^*}{\partial\eta} = \sqrt{\frac{p\lambda}{4(1+\eta)}} > 0$$

The derivatives indicate that given λ and η, the agent will increase the μ when the penalty rate increases. Note that $\sqrt{(1+\eta)p\lambda}-\lambda$, as a function of λ, increases when $(1+\eta)p/4 > \lambda$. From (4.9) and (4.10) the agent installs service capacity $\sqrt{(1+\eta)p\lambda}-\lambda$ when $p \geq p_3$, and from Lemma 4.3 we have $p_3 > 4p_1$. Therefore we have $p > 4p_1 = 4\lambda/(1+\eta) \Rightarrow (1+\eta)p/4 > \lambda \Rightarrow \partial\mu^*/\partial\lambda > 0$. Thus, given p and η, the agent will increase μ when λ increases. Given p and λ, the agent will increase his μ when he is more risk-averse.

The agent's optimal expected utility rate when installing capacity $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda}-\lambda$ is $u_A^* \equiv u_A(\mu^*(w,p);w,p) = w-2\sqrt{(1+\eta)p\lambda}+\lambda$, and it depends on w, p, η and λ. Note that $\partial u_A^*/\partial w = -1 < 0$, $\partial u_A^*/\partial p = -\sqrt{(1+\eta)\lambda/p} < 0$ and $\partial u_A^*/\partial\eta = -\sqrt{p\lambda/(1+\eta)} < 0$, indicating that the agent's optimal expected utility rate decreases with the compensation rate, the penalty rate and his risk intensity. Note that $\partial u_A^*/\partial\lambda = -\left(\sqrt{p}-\sqrt{p_1}\right)/\sqrt{p_1}$, and from Proposition 4.9 $p \geq p_3 > p_1 \Rightarrow \sqrt{p}-\sqrt{p_1} > 0$, therefore the agent's optimal expected utility rate also decreases with the failure rate.

Summary: Recall that given the set of offers $\{(w,p) : p \in (0,\lambda], w \geq p\}$ a risk-neutral agent would accept the contract, install $\mu^*(w,p) = 0$ and receive expected utility rate $u(\mu^*(w,p);w,p) = w-p$. Given the set of offers $\left\{(w,p) : p > \lambda, w \geq 2\sqrt{p\lambda}-\lambda\right\}$ he would accept the contract, install $\mu^*(w,p) = \sqrt{p\lambda}-\lambda$ and receive expected utility rate $u(\mu^*(w,p);w,p) = w-2\sqrt{p\lambda}+\lambda$. By comparing the optimal capacities of a weakly risk-averse agent to that of a risk-neutral agent, three conclusions are drawn:

1. Given a λ, the principal has to set a higher p in order to induce a weakly risk-averse agent to install a positive service capacity versus a risk-neutral agent ($p > \lambda$ for risk-neutral agent, $p > \lambda/(1-\eta)$ for weakly risk-averse agent).
2. Given a λ, when p is relatively low, the μ value plays a more prominent role in the utility of a weakly risk-averse agent who therefore installs a service capacity lower than a risk-neutral agent ($\sqrt{p\lambda}-\lambda > \sqrt{(1-\eta)p\lambda}-\lambda$). As the p increases, the penalty charge and the risk become of greater concern, therefore the weakly risk-averse agent installs a μ^* higher than a risk-neutral agent ($\sqrt{(1+\eta)p\lambda}-\lambda > \sqrt{p\lambda}-\lambda$).
3. In essence, weakly risk-averse attitude makes an agent worse off. We state this conclusion formally in Proposition 4.10.

Proposition 4.10. *Given w and p, an agent who accepts the contract and installs a positive service capacity has a decreasing expected utility rate in $\eta \in [0,4/5)$.*

Proof. Recall that when w and p satisfy conditions (4.8) and (4.9), the agent installs capacity $\mu^*(w,p) = \sqrt{(1-\eta)p\lambda}-\lambda > 0$, and the agent's expected utility rate is $u\left(\mu^*(w,p)\right) = w-\eta p-2\sqrt{(1-\eta)p\lambda}+\lambda$. Note that $\partial u/\partial\eta = -p+p\lambda/\sqrt{(1-\eta)p\lambda} = -\left(p-\sqrt{\lambda/(1-\eta)}\sqrt{p}\right) = -\sqrt{p}\left(\sqrt{p}-\sqrt{p_2}\right)$. Since $p > p_2$, therefore $\partial u/\partial\eta < 0$. When the compensation rate w and the penalty rate p satisfy conditions (4.9) and (4.10), the agent installs capacity $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda > 0$, and the agent's expected utility rate is $u(\mu^*(w,p)) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$, therefore $\partial u / \partial \eta = -\sqrt{p\lambda/(1+\eta)} < 0$. □

The following Corollary follows from Proposition 4.10.

Corollary 4.11. *Given w and p, an agent who accepts the contract and subsequently installs a positive service capacity is always worse off when he is weakly risk-averse ($\eta \in (0, 4/5)$) than risk-neutral ($\eta = 0$).*

We discuss the case for $\eta \geq 4/5$ in Sect. 4.2.1.

4.1.2 Principal's Optimal Strategy

Anticipating the agent's optimal $\mu^*(w,p)$ the principal chooses the w and p that maximize her expected profit rate by solving the optimization problem (4.11).

$$\max_{w>0,p>0} \Pi_P(w,p;\mu^*(w,p)) = \max_{w>0,p>0} \left\{ \frac{r\mu^*(w,p)}{\lambda + \mu^*(w,p)} - w + \frac{p\lambda}{\lambda + \mu^*(w,p)} \right\} \tag{4.11}$$

Denote $(w^*, p^*) = \text{argmax}_{w>0,p>0} \Pi_P(w,p;\mu^*(w,p))$.

Before deriving the principal's optimal strategy, we examine the case when the principal's contract offer satisfies $p = p_3$ and $w \geq w_3$, in which case the agent is indifferent with respect to installing two different service capacities. Nevertheless, the corresponding solutions $((w,p),\mu)$ have to be admissible solutions (see Definition 2.3). We state this case formally in Proposition 4.12.

Proposition 4.12. *Suppose a weakly risk-averse agent. Assume that the principal's potential offers are in the set $\{(w,p) : p = p_3, w \geq w_3\}$.*

(a) *If $r \in (0, p_3)$, the agent installs $\mu^* = \sqrt{(1-\eta)p_3\lambda} - \lambda$ if offered a contract.*
(b) *If $r = p_3$, both $\mu^* = \sqrt{(1-\eta)p_3\lambda} - \lambda$ and $\mu^* = \sqrt{(1+\eta)p_3\lambda} - \lambda$ lead to admissible solutions. Therefore the agent installs either $\sqrt{(1-\eta)p_3\lambda} - \lambda$ or $\sqrt{(1+\eta)p_3\lambda} - \lambda$ if offered a contract.*
(c) *If $r > p_3$, the agent installs $\mu^* = \sqrt{(1+\eta)p_3\lambda} - \lambda$ if offered a contract.*

Proof. Note that for $w \geq w_3$ we have $\partial \Pi_P(w,p_3;\mu)/\partial\mu = (r-p_3)\lambda/(\lambda+\mu)^2$. Define $\mu_L \equiv \sqrt{(1-\eta)p_3\lambda} - \lambda$ and $\mu_H \equiv \sqrt{(1+\eta)p_3\lambda} - \lambda$. Note that $\mu_H > \mu_L$. If $r \in (0,p_3)$, then $\partial\Pi_P/\partial\mu < 0$, therefore $((w,p_3),\mu_L) \succeq ((w,p_3),\mu_H)$. If the principal offers a contract (the conditions are discussed in Proposition 4.18 that follows), then by Definition 2.3 only μ_L leads to admissible solutions. Thus we obtain (a). If $r > p_3$, then $\partial\Pi_P/\partial\mu > 0$, therefore $((w,p_3),\mu_H) \succeq ((w,p_3),\mu_L)$. If the principal offers a contract (see Proposition 4.18), then only μ_H leads to admissible solutions. Therefore we obtain (c). If $r = p_3$, then $\partial\Pi_P/\partial\mu = 0$, indicating that the principal receives the same expected profit rate when the agent installs capacity μ_L or μ_H. If the principal offers a contract (see Proposition 4.18), then both μ_L and μ_H lead to admissible solutions and we obtain (b). $\square$

Notation:

$$r_1 \equiv \eta p_3 + (1-\eta)\sqrt{p_2 p_3}\,,\; r_2 \equiv \left(1 + 2\eta\left(\frac{\sqrt{p_3}-\sqrt{p_2}}{\sqrt{p_2}}\right)\right)p_3,$$
$$\text{and } r_3 \equiv (1+2\eta)p_3 \tag{4.12}$$

Note that r_1, r_2 and r_3 are functions of λ and η. However we suppress the parameters (λ, η).

Define p_{cu} as follows[1]:

$$p_{cu} \equiv \frac{1}{9a^2}\left(b + C + \overline{C}\right)^2 \tag{4.13}$$

where $a \equiv 2\eta$, $b \equiv (1-2\eta)\sqrt{p_2}$, and $d \equiv -r\sqrt{p_2}$ and

$$C \equiv \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}, \overline{C} \equiv \sqrt[3]{\frac{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}},$$
$$\text{where } \Delta_0 \equiv b^2,\ \Delta_1 \equiv 2b^3 + 27a^2 d$$

Replacing Δ_0 and Δ_1 by the expressions of a, b and d we have

$$C = \sqrt[3]{\frac{2(1-2\eta)^3\sqrt{p_2^3} - 108\eta^2 r\sqrt{p_2} + \sqrt{-432\eta^2 r(1-2\eta)^3 p_2^2 + 11664\eta^4 r^2 p_2}}{2}}$$

and

$$\overline{C} = \sqrt[3]{\frac{2(1-2\eta)^3\sqrt{p_2^3} - 108\eta^2 r\sqrt{p_2} - \sqrt{-432\eta^2 r(1-2\eta)^3 p_2^2 + 11664\eta^4 r^2 p_2}}{2}}$$

We introduce several technical lemmas with proofs in the Appendix.

Lemma 4.13. *Let* $4/5 > \eta > 0$ *and* $\lambda > 0$.

(a) $\eta p_3 + (1-\eta)\sqrt{p_2 p_3} > p_2 > 0$.
(b) $p_3 > \eta p_3 + (1-\eta)\sqrt{p_2 p_3}$.
(c) $\left(1 + 2\eta\left(\sqrt{p_3} - \sqrt{p_2}\right)/\sqrt{p_2}\right)p_3 > p_3$.
(d) $(1+2\eta)p_3 > \left(1 + 2\eta\left(\sqrt{p_3} - \sqrt{p_2}\right)/\sqrt{p_2}\right)p_3$.

Lemma 4.13 implies that for $\eta \in (0, 4/5)$ we have $r_3 > r_2 > p_3 > r_1 > p_2$.

[1] The subscript "*cu*" stands for "cubic" because (4.13) is the square of the solution to Eq. (A.1), which is a cubic equation that is introduced later in the proof for Lemma 4.16.

Lemma 4.14. *Let* $\eta > 0$ *and* $\lambda > 0$.

(a) *If* $\left(1+3\eta+2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta) > r > \left(1+3\eta-2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta)$ *then* $0 > r-2\sqrt{(1+2\eta)r\lambda/(1+\eta)}+\lambda$.

(b) *If* $\left(1+3\eta-2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta) > r > 0$ *or* $r > \left(1+3\eta+2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta)$ *then* $r-2\sqrt{(1+2\eta)r\lambda/(1+\eta)}+\lambda > 0$.

(c) *If* $r = \left(1+3\eta-2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta)$ *or* $r = \left(1+3\eta+2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta)$ *then* $r-2\sqrt{(1+2\eta)r\lambda/(1+\eta)}+\lambda = 0$.

Lemma 4.15. *Given* $1 > \eta > 0$ *and* $\lambda > 0$, *then* $(1+2\eta)p_3 > \left(1+3\eta+2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta)$.

Lemma 4.16. *Consider* $\max_{x\in[\sqrt{p_2},\sqrt{p_3}]} f(x)$ *where* $f(x) = r+\lambda-\eta x^2-\sqrt{p_2}\,((1-2\eta)x+r/x)$ *and denote* $x^* = \operatorname{argmax}_{x\in[\sqrt{p_2},\sqrt{p_3}]} f(x)$. *The solutions to this optimization problem are*

(a) $x^* = \sqrt{p_2}$ *if* $r \in (0, p_2]$.
(b) $x^* = \sqrt{p_{cu}} \in \left(\sqrt{p_2}, \sqrt{p_3}\right)$ *if* $r \in (p_2, r_2)$.
(c) $x^* = \sqrt{p_3}$ *if* $r \geq r_2$.

Lemma 4.17. *Consider* $\max_{x\geq\sqrt{p_3}} f(x)$ *where* $f(x) = r+\lambda-\sqrt{p_1}\,((1+2\eta)x+r/x)$ *and denote* $x^* = \operatorname{argmax}_{x\geq\sqrt{p_3}} f(x)$. *The solutions to this optimization problem are*

(a) $x^* = \sqrt{p_3}$ *if* $r \in (0, r_3]$.
(b) $x^* = \sqrt{r/(1+2\eta)}$ *if* $r > r_3$.

Now we state Proposition 4.18, which serves as a stepping stone towards the main results Theorem 4.19 and Proposition 4.20 that follow later. Proposition 4.18 provides the optimal w^* and optimal p^* under some restrictions. These restrictions are later removed in the main results Theorem 4.19 and Proposition 4.20.

Recall that Proposition 4.9 describes the agent's optimal response to each contract offer (w, p). Since the principal will not propose a contract that is going to be rejected by a weakly risk-averse (WRA) agent, therefore Proposition 4.18 only considers pairs (w, p) that result in agent's non-negative expected utility rate. Define:

$$\begin{aligned}
\mathfrak{D}_{(4.7)} &\equiv \{(w,p) \text{ that satisfies (4.7) when } \eta \in (0,4/5)\}\\
\mathfrak{D}_{(4.8)} &\equiv \{(w,p) \text{ that satisfies (4.8) when } \eta \in (0,4/5)\}\\
\mathfrak{D}_{(4.9)} &\equiv \{(w,p) \text{ that satisfies (4.9) when } \eta \in (0,4/5)\}\\
\mathfrak{D}_{(4.10)} &\equiv \{(w,p) \text{ that satisfies (4.10) when } \eta \in (0,4/5)\}\\
\mathfrak{D}_{\text{WRA}} &\equiv \mathfrak{D}_{(4.7)} \cup \mathfrak{D}_{(4.8)} \cup \mathfrak{D}_{(4.9)} \cup \mathfrak{D}_{(4.10)}
\end{aligned} \tag{4.14}$$

Proposition 4.18. *Given a weakly risk-averse agent;*

(a) *If* $(w,p) \in \mathfrak{D}_{(4.7)}$, *then the principal does not propose a contract.*
(b) *Consider offers* $(w,p) \in \mathfrak{D}_{(4.8)} \cup \mathfrak{D}_{(4.9)}$.

(b1) *If* $r \in (0,p_2]$, *then the principal does not propose a contract.*
(b2) *If* $r \in (p_2,p_3]$, *then the principal offers* $(w^*,p^*) = \left(\eta p_{cu} + 2\sqrt{(1-\eta)p_{cu}\lambda} - \lambda, p_{cu}\right)$ *and the agent installs service capacity* $\mu^*(w^*,p^*) = \sqrt{(1-\eta)p_{cu}\lambda} - \lambda$.
(b3) *If* $r \in (p_3,r_2)$, *then the principal either offers* $(w^*,p^*) = (w_3,p_3)$ *and the agent installs* $\mu^*(w,p) = \sqrt{(1+\eta)p_3\lambda} - \lambda$, *or offers* $(w^*,p^*) = \left(\eta p_{cu} + 2\sqrt{(1-\eta)p_{cu}\lambda} - \lambda, p_{cu}\right)$ *and the agent installs service capacity* $\mu^*(w^*,p^*) = \sqrt{(1-\eta)p_{cu}\lambda} - \lambda$.
(b4) *If* $r \geq r_2$, *then the principal's offer is* $(w^*,p^*) = (w_3,p_3)$ *and the agent installs service capacity* $\mu^*(w^*,p^*) = \sqrt{(1+\eta)p_3\lambda} - \lambda$.

(c) *Consider offers* $(w,p) \in \mathfrak{D}_{(4.9)} \cup \mathfrak{D}_{(4.10)}$.

(c1) *If* $r \in (0,r_1]$, *then the principal does not propose a contract.*
(c2) *If* $r \in (r_1,p_3]$, *the principal offers a contract with* $(w^*,p^*) = (w_3,p_3)$ *and the agent installs service capacity* $\mu^*(w^*,p^*) = \sqrt{(1-\eta)p_3\lambda} - \lambda$.
(c3) *If* $r \in (p_3,r_3]$, *the principal offers a contract with* $(w^*,p^*) = (w_3,p_3)$ *and the agent installs service capacity* $\mu^*(w^*,p^*) = \sqrt{(1+\eta)p_3\lambda} - \lambda$.
(c4) *If* $r > r_3$, *the principal offers* $(w^*,p^*) = \left(2\sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda, r/(1+2\eta)\right)$ *and the agent installs service capacity* $\mu^*(w^*,p^*) = \sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$.

Proof. The structure of the proof for Proposition 4.18 is depicted in Fig. 4.7.

Case $(w,p) \in \mathfrak{D}_{(4.7)}$: According to Proposition 4.9 part (a), in case the principal makes an offer, the agent accepts the contract but does not install any service capacity. Since $\partial\Pi_P/\partial w = -1 < 0$, thus $w^* = p$ and from Eq. (3.3) $\Pi_P(w^*,p;\mu^*(w^*,p)) = -w^* + p = -p + p = 0$. Therefore the principal does not propose a contract.

Case $(w,p) \in \mathfrak{D}_{(4.8)} \cup \mathfrak{D}_{(4.9)}$: According to Proposition 4.9 part (b), if $(w,p) \in \mathfrak{D}_{(4.8)}$, then in case the principal makes an offer, the agent accepts the contract and installs $\sqrt{(1-\eta)p\lambda} - \lambda$. Since $\partial\Pi_P/\partial w = -1 < 0$, therefore $w^* = \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda$. According to Propositions 4.9 part (c) and 4.12, if $(w,p) \in \mathfrak{D}_{(4.9)}$ (which implies $p = p_3$), then in case the principal makes an offer, the agent accepts the contract and installs $\sqrt{(1-\eta)p_3\lambda} - \lambda$ if $r \in (0,p_3)$, installs either $\sqrt{(1-\eta)p_3\lambda} - \lambda$ or $\sqrt{(1+\eta)p_3\lambda} - \lambda$ if $r = p_3$, or installs $\sqrt{(1+\eta)p_3\lambda} - \lambda$ if $r > p_3$. Since $\partial\Pi_P/\partial w = -1 < 0$, therefore $w^* = w_3$. Denote the principal's expected profit rate when $(w,p) = (w_3,p_3)$ and $\mu = \sqrt{(1-\eta)p_3\lambda} - \lambda$ by

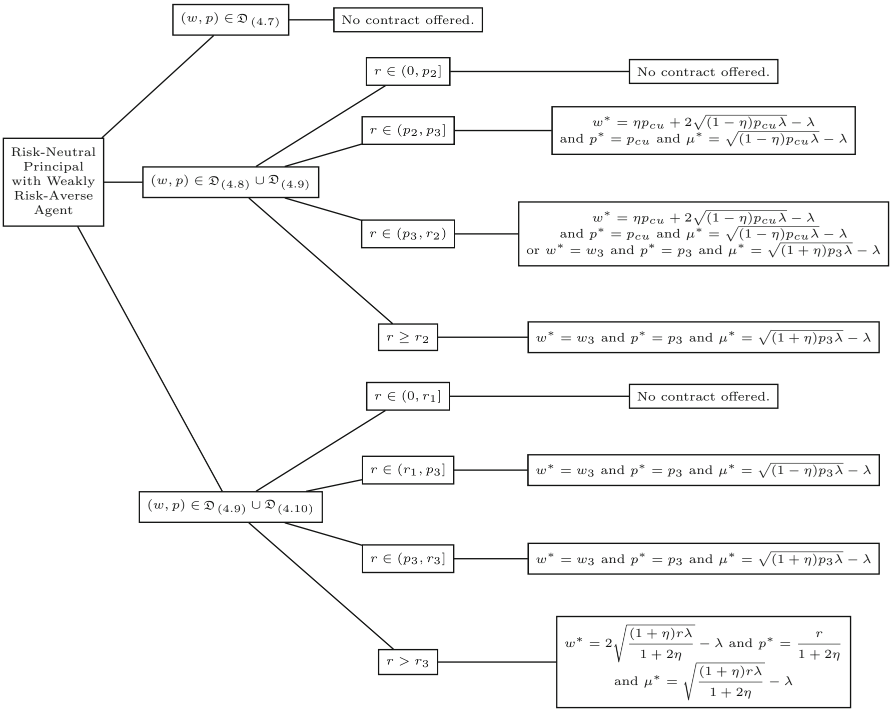


Fig. 4.7 Structure of the proof for Proposition 4.18

$\Pi_P^L(p_3)$, and denote the principal's expected profit rate when $(w,p) = (w_3, p_3)$ and $\mu = \sqrt{(1+\eta)p_3\lambda} - \lambda$ by $\Pi_P^H(p_3)$. By plugging the value of w, p and μ into Eq. (3.3):

$$\Pi_P^L(p_3) = r + \lambda - \eta p_3 - \sqrt{p_2}\left((1-2\eta)\sqrt{p_3} + \frac{r}{\sqrt{p_3}}\right)$$

$$= \left(\frac{\sqrt{p_3} - \sqrt{p_2}}{\sqrt{p_3}}\right)(r - r_1) \tag{4.15}$$

$$\Pi_P^H(p_3) = r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p_3} + \frac{r}{\sqrt{p_3}}\right) \tag{4.16}$$

and the principal's optimization problem is $\max_{p \in [p_2, p_3]} \Pi_P(w^*, p; \mu^*(w^*, p))$ where:

$$\Pi_P(w^*, p; \mu^*(w^*, p)) = \begin{cases} r + \lambda - \eta p - \sqrt{p_2}\left((1-2\eta)\sqrt{p} + \frac{r}{\sqrt{p}}\right), \\ \quad \text{for } p \in [p_2, p_3) \\ \max\left\{\Pi_P^L(p_3), \Pi_P^H(p_3)\right\}, \text{ for } p = p_3 \end{cases}$$

Define $x \equiv \sqrt{p}$, the expression $r + \lambda - \eta p - \sqrt{p_2}\left((1-2\eta)\sqrt{p} + r/\sqrt{p}\right)$ can be restated as $f(x) = r + \lambda - \eta x^2 - \sqrt{p_2}\,((1-2\eta)x + r/x)$. Maximizing $f(x)$ with respect to x over $\left[\sqrt{p_2}, \sqrt{p_3}\right]$ is equivalent to maximizing $r + \lambda - \eta p - \sqrt{p_2}\left((1-2\eta)\sqrt{p} + r/\sqrt{p}\right)$ with respect to p over the interval $[p_2, p_3]$ in the sense that

$$\underset{p\in[p_2,p_3]}{\operatorname{argmax}}\left\{r + \lambda - \eta p - \sqrt{p_2}\left((1-2\eta)\sqrt{p} + \frac{r}{\sqrt{p}}\right)\right\} = \left(\underset{x\in\left[\sqrt{p_2},\sqrt{p_3}\right]}{\operatorname{argmax}}\ f(x)\right)^2$$

According to Lemma 4.13, $r_2 > p_3 > p_2$, therefore we examine the following subcases.

Subcase $r \in (0, p_2]$: According to Lemma 4.16 part (a), $p^* = p_2$; this case is taken care of in the case when $(w, p) \in \mathfrak{D}_{(4.7)}$ and the principal does not propose a contract.

Subcase $r \in (p_2, p_3]$: According to Lemma 4.16 part (b) and Proposition 4.12 part (a) and (b), $p^* = p_{cu}$ and the principal's expected profit rate is $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) > \Pi_P(p_2, p_2; 0) = 0$. Thus the principal proposes $w^* = \eta p_{cu} + 2\sqrt{(1-\eta)p_{cu}\lambda} - \lambda$ and $p^* = p_{cu}$ that induces the agent to install $\mu^*(w^*, p^*) = \sqrt{(1-\eta)p_{cu}\lambda} - \lambda$.

Subcase $r \in (p_3, r_2)$: According to Lemma 4.16 part (b) and Proposition 4.12 part (c), the principal chooses either $p^* = p_{cu}$ with expected profit rate $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r + \lambda - \eta p_{cu} - \sqrt{p_2}\left((1-2\eta)\sqrt{p_{cu}} + r/\sqrt{p_{cu}}\right) > \Pi_P(p_2, p_2; 0) = 0$, or chooses $p^* = p_3$ with expected profit rate $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = \Pi_P^H(p_3) > \Pi_P^L(p_3) = \left(\sqrt{p_3} - \sqrt{p_2}\right)(r - r_1)/\sqrt{p_3} > 0$. However due to the difficulty of computing p_{cu} we do not explicitly identify the principal's optimal offer.

Subcase $r \geq r_2$: According to Lemma 4.16 part (c), $p^* = p_3$. According to Proposition 4.12 part (c) the agent installs capacity $\sqrt{(1+\eta)p_3\lambda} - \lambda$ and the principal's expected profit rate is $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = \Pi_P^H(p_3) > \Pi_P^L(p_3) = \left(\sqrt{p_3} - \sqrt{p_2}\right)(r - r_1)/\sqrt{p_3} > 0$. Therefore the principal proposes $w^* = 2\sqrt{(1+\eta)p_3\lambda} - \lambda$ and $p^* = p_3$ that induces the agent to install $\mu^*(w^*, p^*) = \sqrt{(1+\eta)p_3\lambda} - \lambda$.

Case $(w, p) \in \mathfrak{D}_{(4.9)} \cup \mathfrak{D}_{(4.10)}$: According to Proposition 4.9 part (d), if $(w, p) \in \mathfrak{D}_{(4.10)}$, then in case the principal makes an offer, the agent accepts the contract and installs $\sqrt{(1+\eta)p\lambda} - \lambda$. Since $\partial\Pi_P/\partial w = -1 < 0$, therefore $w^* = 2\sqrt{(1+\eta)p\lambda} - \lambda$. According to Propositions 4.9 part (c) and 4.12, if $(w, p) \in \mathfrak{D}_{(4.9)}$ (which implies $p = p_3$), then in case the principal makes an offer, the agent accepts the contract and installs $\sqrt{(1-\eta)p_3\lambda} - \lambda$ if $r \in (0, p_3)$, installs either $\sqrt{(1-\eta)p_3\lambda} - \lambda$ or $\sqrt{(1+\eta)p_3\lambda} - \lambda$ if $r = p_3$, or installs $\sqrt{(1+\eta)p_3\lambda} - \lambda$ if $r > p_3$. Since $\partial\Pi_P/\partial w = -1 < 0$, therefore $w^* = w_3$. Recall the definition of $\Pi_P^L(p_3)$ and $\Pi_P^H(p_3)$ (see Eqs. (4.15) and (4.16)). The principal's optimization problem is $\max_{p \geq p_3} \Pi_P(w^*, p; \mu^*(w^*, p))$ where:

$$\Pi_P(w^*, p; \mu^*(w^*, p)) = \begin{cases} \max\left\{\Pi_P^L(p_3), \Pi_P^H(p_3)\right\}, \text{ for } p = p_3 \\ r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p} + \dfrac{r}{\sqrt{p}}\right), \text{ for } p > p_3 \end{cases}$$

Define $x \equiv \sqrt{p}$, the expression $r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p} + r/\sqrt{p}\right)$ can be restated as $f(x) = r + \lambda - \sqrt{p_1}\,((1+2\eta)x + r/x)$. Maximizing $f(x)$ for $x \geq \sqrt{p_3}$ is equivalent to maximizing $r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p} + r/\sqrt{p}\right)$ for $p \geq p_3$ in the sense that

$$\underset{p \geq p_3}{\operatorname{argmax}}\left\{r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p} + \frac{r}{\sqrt{p}}\right)\right\} = \left(\underset{x \geq \sqrt{p_3}}{\operatorname{argmax}} f(x)\right)^2$$

According to Lemma 4.13, $r_3 > p_3 > r_1$, therefore we examine the following subcases.

Subcase $r \in (0, r_1]$: According to Lemma 4.17 part (a), $p^* = p_3$. By Proposition 4.12 part (a), $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = \Pi_P^L(p_3) = \left(\sqrt{p_3} - \sqrt{p_2}\right)(r - r_1)/\sqrt{p_3}$ and note that $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) \leq 0$, therefore the principal does not propose a contract.

Subcase $r \in (r_1, p_3]$: According to Lemma 4.17 part (a), $p^* = p_3$. According to Proposition 4.12 part (a) and (b), the principal's expected profit rate is $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = \Pi_P^L(p_3) = \left(\sqrt{p_3} - \sqrt{p_2}\right)(r - r_1)/\sqrt{p_3} > 0$, therefore the principal proposes a contract with $w^* = w_3$ and $p^* = p_3$ that induces the agent to install $\mu^*(w^*, p^*) = \sqrt{(1-\eta)p_3\lambda} - \lambda$.

Subcase $r \in (p_3, r_3]$: According to Lemma 4.17 part (a), $p^* = p_3$. According to Proposition 4.12 part (c), the principal's expected profit rate is $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = \Pi_P^H(p_3) > \Pi_P^L(p_3) = \left(\sqrt{p_3} - \sqrt{p_2}\right)(r - r_1)/\sqrt{p_3} > 0$, therefore the principal proposes a contract with $w^* = w_3$ and $p^* = p_3$ that induces the agent to install $\mu^*(w^*, p^*) = \sqrt{(1+\eta)p_3\lambda} - \lambda$.

Subcase $r > r_3$: According to Lemma 4.17 part (b), $p^* = r/(1+2\eta)$ and the principal's expected profit rate is $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r - 2\sqrt{(1+2\eta)r\lambda/(1+\eta)} + \lambda$. According to Lemmas 4.14 and 4.15, $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) > 0$, therefore the principal proposes a contract with $w^* = 2\sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$ and $p^* = r/(1+2\eta)$ that induces the agent to install service capacity $\mu^*(w^*, p^*) = \sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$.

□

We describe the principal's optimal strategy in Theorem 4.19 and Proposition 4.20. We identify the principal's optimal offer only when $r \in (0, p_3]$ or $r \geq r_2$, (see Theorem 4.19). The cases when $r \in (p_3, r_2)$ are discussed in Proposition 4.20. We prove Theorem 4.19 and Proposition 4.20 together.

Theorem 4.19. *Consider a weakly risk-averse agent and* $(w, p) \in \mathfrak{D}_{\text{WRA}}$.

(a) *If* $r \in (0, p_2]$*, then the principal does not propose a contract.*
(b) *If* $r \in (p_2, p_3]$*, then the principal's offer and the capacity installed by the agent are*

$$(w^*, p^*) = \left(\eta p_{cu} + 2\sqrt{(1-\eta)p_{cu}\lambda} - \lambda, p_{cu}\right) \text{ and}$$
$$\mu^*(w^*, p^*) = \sqrt{(1-\eta)p_{cu}\lambda} - \lambda \tag{4.17}$$

and the principal's expected profit rate is

$$\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r + \lambda - \eta p_{cu} - \sqrt{p_2}\left((1-2\eta)\sqrt{p_{cu}} + \frac{r}{\sqrt{p_{cu}}}\right) \tag{4.18}$$

(c) *If* $r \in [r_2, r_3]$*, then the principal's offer and the capacity installed by the agent are*

$$(w^*, p^*) = (w_3, p_3) \text{ and } \mu^*(w^*, p^*) = \sqrt{(1+\eta)p_3\lambda} - \lambda \tag{4.19}$$

and the principal's expected profit rate is

$$\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p_3} + \frac{r}{\sqrt{p_3}}\right) \tag{4.20}$$

(d) *If* $r > r_3$*, then the principal's offer and the capacity installed by the agent are*

$$(w^*, p^*) = \left(2\sqrt{\frac{(1+\eta)r\lambda}{1+2\eta}} - \lambda, \frac{r}{1+2\eta}\right) \text{ and}$$
$$\mu^*(w^*, p^*) = \sqrt{\frac{(1+\eta)r\lambda}{1+2\eta}} - \lambda \tag{4.21}$$

and the principal's expected profit rate is

$$\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r - 2\sqrt{\frac{(1+2\eta)r\lambda}{1+\eta}} + \lambda \tag{4.22}$$

Proposition 4.20. *Given a weakly risk-averse agent and* $(w, p) \in \mathfrak{D}_{\mathrm{WRA}}$*. If* $r \in (p_3, r_2)$*, then either*

$$(w^*, p^*) = \left(\eta p_{cu} + 2\sqrt{(1-\eta)p_{cu}\lambda} - \lambda, p_{cu}\right) \text{ and}$$
$$\mu^*(w^*, p^*) = \sqrt{(1-\eta)p_{cu}\lambda} - \lambda$$

resulting in principal's expected profit rate

$$\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r + \lambda - \eta p_{cu} - \sqrt{p_2}\left((1-2\eta)\sqrt{p_{cu}} + r/\sqrt{p_{cu}}\right)$$

or the principal offers and the agent installs

$$(w^*, p^*) = (w_3, p_3) \text{ and } \mu^*(w^*, p^*) = \sqrt{(1+\eta)p_3\lambda} - \lambda$$

resulting in principal's expected utility rate

$$\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p_3} + r/\sqrt{p_3}\right)$$

Proof. In part (b) of Proposition 4.18, we solved for (w^*, p^*) by restricting r to be in $(0, p_2]$, or in $(p_2, p_3]$, or in (p_3, r_2), or in $[r_2, +\infty)$. In part (c) of Proposition 4.18, we solved for (w^*, p^*) by restricting r to be in $(0, r_1]$, or in $(r_1, p_3]$, or in $(p_3, r_3]$, or in $(r_3, +\infty)$. The principal maximizes her expected profit rate by offering contract that lead to admissible solutions (Definition 2.3) for any given value of r, η and λ. The structure of the proof for Theorem 4.19 and Proposition 4.20 is depicted in Fig. 4.8.

Case $r \in (0, p_2]$: According to Proposition 4.18 part (a), (b1) and (c1), the principal does not propose a contract. This case corresponds to Theorem 4.19 (a).

Case $r \in (p_2, p_3]$: If $r \in (p_2, r_1]$, then according to Proposition 4.18 part (a), (b2) and (c1), the principal offers $(w^*, p^*) = \left(\eta p_{cu} + 2\sqrt{(1-\eta)p_{cu}\lambda} - \lambda, p_{cu}\right)$

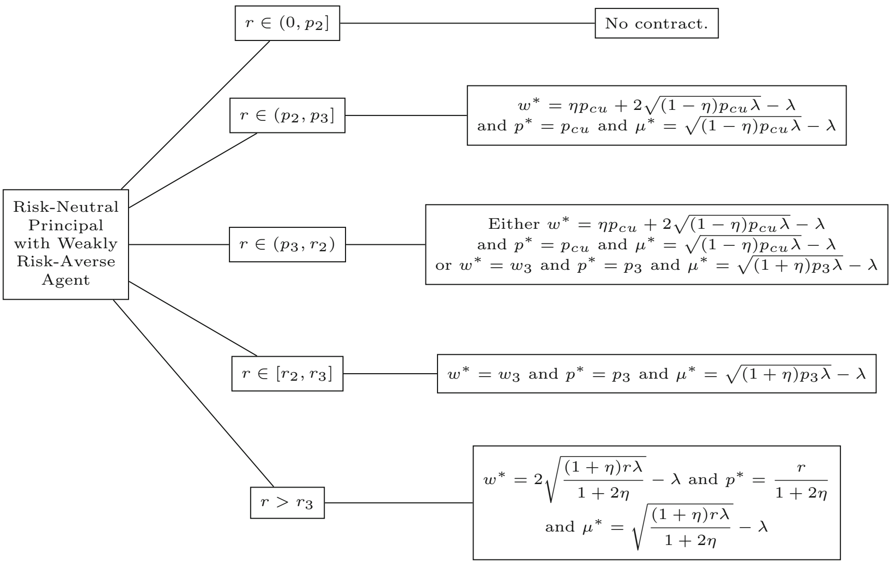


Fig. 4.8 Structure of the proof for Theorem 4.19 and Proposition 4.20

and the agent installs $\mu^*(w^*,p^*) = \sqrt{(1-\eta)p_{cu}\lambda} - \lambda$. If $r \in (r_1, p_3]$, then according to Proposition 4.18 part (a), (b2) and (c2) and Lemma 4.16 part (b) the principal offers $(w^*,p^*) = \left(\eta p_{cu} + 2\sqrt{(1-\eta)p_{cu}\lambda} - \lambda, p_{cu}\right)$ and the agent installs $\mu^*(w^*,p^*) = \sqrt{(1-\eta)p_{cu}\lambda} - \lambda$. This case is addressed in Theorem 4.19 (b).

Case $r \in (p_3, r_2)$: According to Proposition 4.18 part (a), (b3) and (c3), the principal either offers a contract $(w^*,p^*) = \left(\eta p_{cu} + 2\sqrt{(1-\eta)p_{cu}\lambda} - \lambda, p_{cu}\right)$ and the agent installs $\mu^*(w^*,p^*) = \sqrt{(1-\eta)p\lambda} - \lambda$, or offers $(w^*,p^*) = (w_3, p_3)$ and the agent installs $\mu^*(w^*,p^*) = \sqrt{(1+\eta)p_3\lambda} - \lambda$. This case corresponds to Proposition 4.20.

Case $r \in [r_2, r_3]$: According to Proposition 4.18 part (a), (b4) and (c3), the principal offers a contract with $(w^*,p^*) = (w_3, p_3)$ and the agent installs $\mu^*(w^*,p^*) = \sqrt{(1+\eta)p_3\lambda} - \lambda$. This case corresponds to Theorem 4.19 (c).

Case $r > r_3$: According to Proposition 4.18 part (a), (b4) and (c4) and Lemma 4.17 part (b), the principal offers a contract with $(w^*,p^*) = \left(2\sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda, r/(1+2\eta)\right)$ and the agent installs service capacity $\mu^*(w^*,p^*) = \sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$. This case corresponds to Theorem 4.19 (d).

□

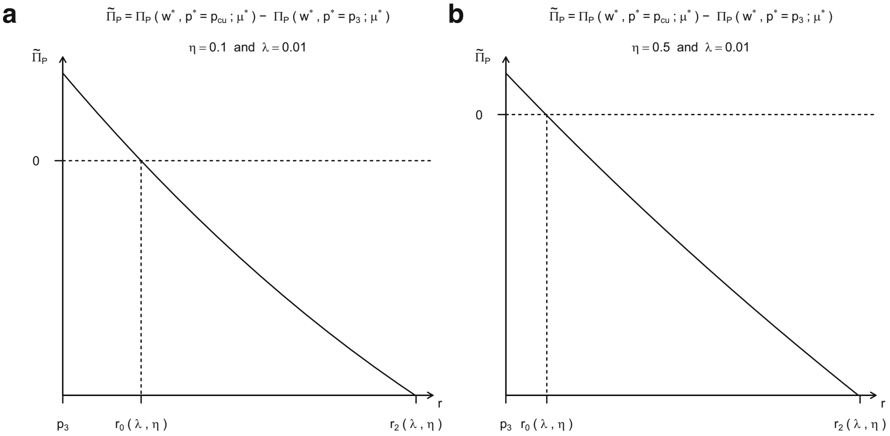
Theorem 4.19 and Proposition 4.20 indicate that the existence of a beneficial contract with a weakly risk-averse agent is determined exogenously by the revenue rate r, the failure rate λ, and the risk coefficient η.

Since it is difficult to identify the principal's optimal offer when $r \in (p_3, r_2)$ due to the difficulty of computing p_{cu} we resort to numerical results to better understand the principal's choices.

Remark 4.21. Figure 4.9 demonstrates that when $\eta = 0.1$ and $\eta = 0.5$ there exists an $r_0 \in (p_3, r_2)$ such that when $r \in (p_3, r_0)$, the principal offers $(w^*,p^*) = \left(\eta p_{cu} + 2\sqrt{(1-\eta)p_{cu}\lambda} - \lambda, p_{cu}\right)$, when $r \in (r_0, r_2)$, she offers $(w^*,p^*) = (w_3, p_3)$ and when $r = r_0$, the principal is indifferent about the two alternative offers. However due to the difficulty of computing p_{cu} (Eq. (4.13)), it is not clear how to determine the general existence of such an r_0 for all $\eta \in (0, 4/5)$ and identify an explicit expression of r_0 as a function of λ and η.

4.2 Optimal Strategies Given a Strongly Risk-Averse Agent

For the strongly risk-averse (SRA) agent we first derive the agent's optimal strategy. The agent's optimization problem is stated in (4.4).



$\eta = 0.1$ $\eta = 0.5$

Fig. 4.9 The value of $\tilde{\Pi}_P \equiv \Pi_P(w^*, p^* = p_{cu}; \mu^*) - \Pi_P(w^*, p^* = p_3; \mu^*)$ for $r \in (p_3, r_2)$

Notation:

$$w_4 = p_4 \equiv \left(1 + 2\eta + 2\sqrt{\eta(1+\eta)}\right)\lambda \tag{4.23}$$

Note that w_4 and p_4 are functions of λ and η which are suppressed in our notation.

A technical lemma used later is introduced next (see proof in the Appendix).

Lemma 4.22. *Let* $\eta > 1/3$ *and* $\lambda > 0$*, then* $\left(1 + 2\eta + 2\sqrt{\eta(1+\eta)}\right)\lambda > 4\lambda/(1+\eta)$.

We describe a strongly risk-averse agent's optimal response to any possible offered contract $(w, p) \in \mathbb{R}^2_+$ in Proposition 4.23.

Proposition 4.23. *Consider a strongly risk-averse agent* ($\eta \geq 4/5$).

(a) *Given*

$$w \geq p \in (0, p_4) \tag{4.24}$$

then the agent would accept the contract if offered and install $\mu^*(w, p) = 0$ *with resulting expected utility rate* $u_A(\mu^*(w, p); w, p) = w - p \geq 0$. *The agent rejects the contract if* $p \in (0, p_4)$ *and* $w \in (0, p)$.

(b) *Given*

$$p = p_4 \text{ and } w \geq w_4 \tag{4.25}$$

then the agent would accept the contract if offered and is indifferent about installing either $\mu^*(w,p) = 0$ *or* $\mu^*(w,p) = \sqrt{(1+\eta)p_4\lambda} - \lambda$*. In both cases the agent's expected utility rate is* $u_A(\mu^*(w,p); w,p) = w - p_4 \geq 0$*. If* $r \in (0, p_4]$*, then neither* $\mu^* = 0$ *nor* $\mu^* = \sqrt{(1+\eta)p_4\lambda} - \lambda$ *leads to admissible solutions (see Definition 2.3). If* $r > p_4$*, then there exists* w^* *such that* $\left((w^*, p_4), \mu^* = \sqrt{(1+\eta)p_4\lambda} - \lambda\right)$ *is the unique admissible solution (for proof see Proposition 4.24). The agent rejects the contract if* $p = p_4$ *and* $w \in (0, w_4)$*.*

(c) *Given*

$$p > p_4 \text{ and } w \geq 2\sqrt{(1+\eta)p\lambda} - \lambda \tag{4.26}$$

then the agent would accept the contract if offered and install $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ *with resulting expected utility rate* $u_A(\mu^*(w,p); w,p) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda \geq 0$*. The agent rejects the contract if* $p > p_4$ *and* $w \in \left(0, 2\sqrt{(1+\eta)p\lambda} - \lambda\right)$*.*

Proof. According to Table 4.1, the optimization of $u(\mu)$ when $\eta \in [4/5, 1)$ versus $\eta \geq 1$ is different. Therefore we prove the proposition separately for $\eta \in [4/5, 1)$ and $\eta \geq 1$.

Case $\eta \in [4/5, 1)$: Recall the definition of p_1 and p_2 in (4.5). Note that $4p_2 > p_2 > 4p_1$ and according to Lemmas 4.7 part (b) and (c) and 4.22, $p_2 \geq p_4 > 4p_1$. Therefore we have $4p_2 > p_2 \geq p_4 > 4p_1$. Figure 4.10 depicts the shape of $u(\mu)$ when $\eta \in [4/5, 1)$ and the value of p falls in different ranges. The structure of the proof when $\eta \in [4/5, 1)$ is depicted in Fig. 4.11.

Case $p \in (0, 4p_1]$: According to Table 4.1, $u(\mu)$ is decreasing for $\mu \geq 0$. Thus the agent's optimal service capacity is $\mu^*(w,p) = 0$ and from (4.3) $u(\mu^*(w,p)) = w - p$.

Subcase $w \in (0, p)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.
Subcase $w \geq p$: $u(\mu^*(w,p)) \geq 0$, thus the agent would accept the contract if offered.

Case $p \in (4p_1, p_2]$: According to Table 4.1, there is a service capacity that maximizes $u(\mu)$ for $\mu \in [0, \lambda)$ and a service capacity that maximizes $u(\mu)$ for $\mu > \lambda$. Denote the optimal service capacity in $[0, \lambda)$ by $\mu^*_{[0,\lambda)}(w,p)$. Note that $u(\mu)$ is decreasing with respect to μ over $[0, \lambda)$, therefore the agent's optimal service capacity is $\mu^*_{[0,\lambda)}(w,p) = 0$ and from (4.3) $u\left(\mu^*_{[0,\lambda)}(w,p)\right) = w - p$. Denote the optimal service capacity for $\mu > \lambda$ by $\mu^*_\lambda(w,p)$. From the first order condition $\mu^*_\lambda(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and from (4.3) $u\left(\mu^*_\lambda(w,p)\right) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. The agent has a choice of two service capacities and he installs the one that generates a higher expected utility rate. Note that $u\left(\mu^*_\lambda(w,p)\right) - u\left(\mu^*_{[0,\lambda)}(w,p)\right) = p - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to

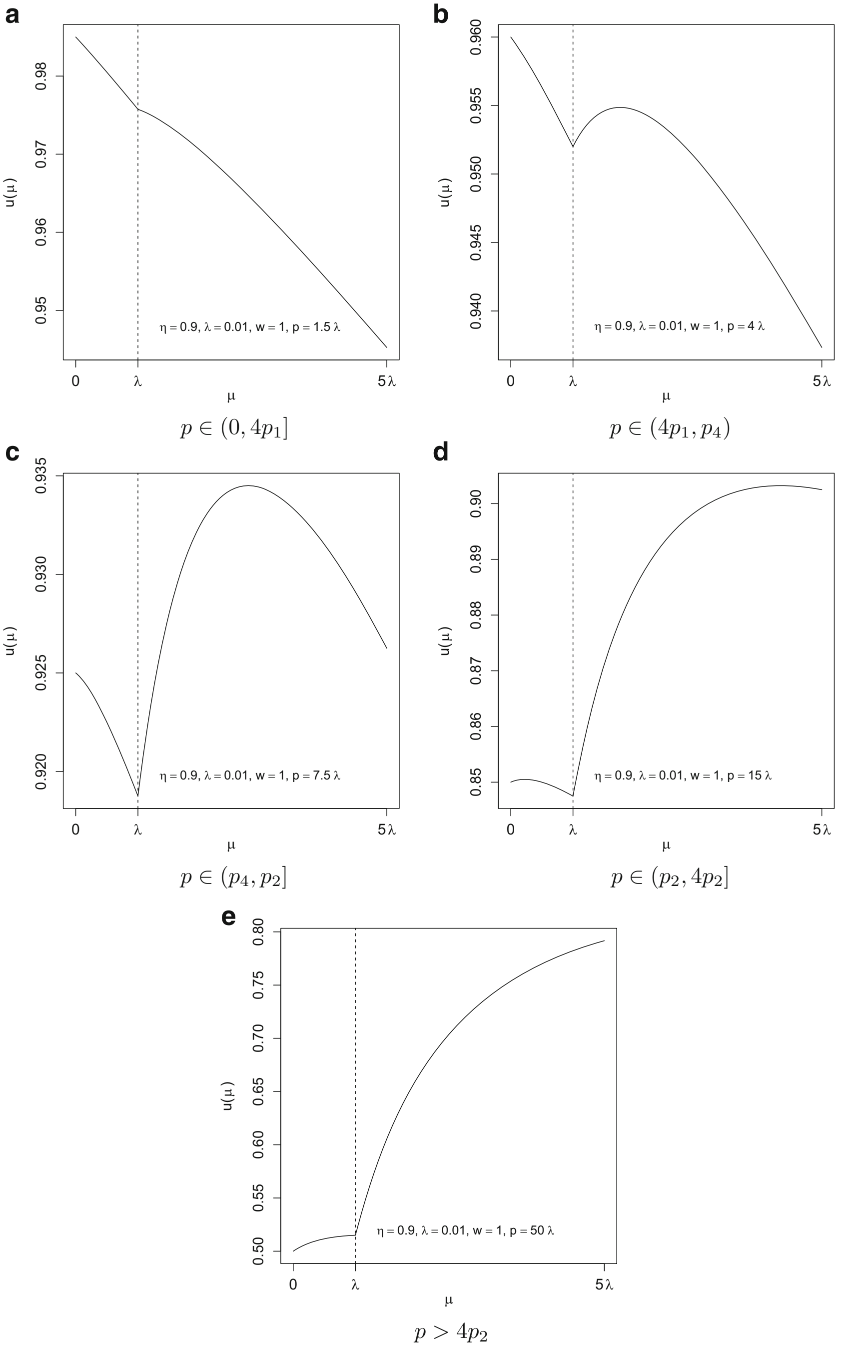


Fig. 4.10 Illustration of the forms of $u(\mu)$ when $\eta \in [4/5, 1)$

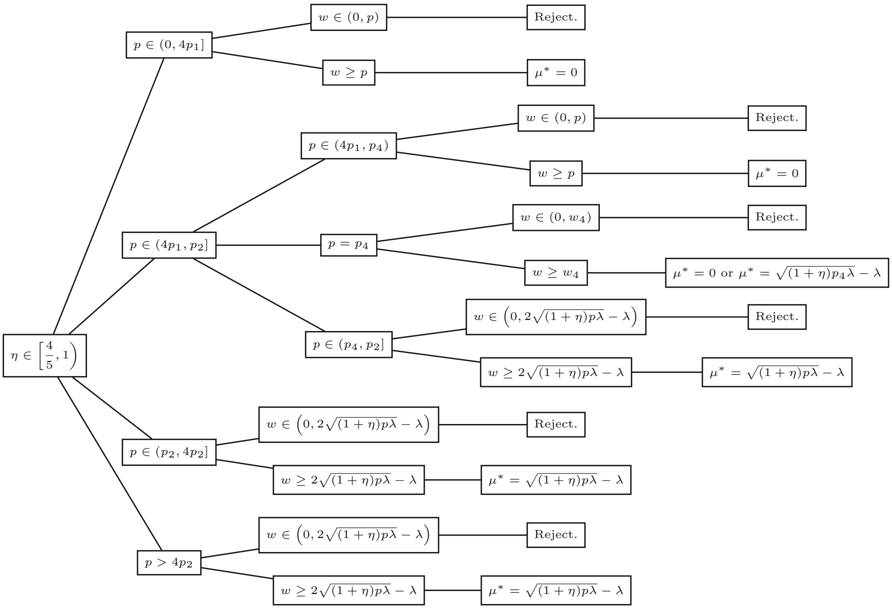


Fig. 4.11 Structure of the proof for Proposition 4.23 when $\eta \in [4/5, 1)$

Lemmas 4.7 part (b) and (c) and 4.22, $p_2 \geq p_4 > 4p_1$, therefore we examine the following subcases.

Subcase $p \in (4p_1, p_4)$: According to Lemma 4.6, $4p_1 > \left(1 + 2\eta - 2\sqrt{\eta(1+\eta)}\right)\lambda$ and according to Lemma 4.5 part (a), $u\left(\mu^*_{[0,\lambda)}(w,p)\right) > u\left(\mu^*_{\lambda}(w,p)\right)$, thus the agent's optimal service capacity is $\mu^*(w,p) = 0$ and $u(\mu^*(w,p)) = w - p$.

Subsubcase $w \in (0, p)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

Subsubcase $w \geq p$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

Subcase $p = p_4$: According to Lemma 4.5 part (c), $u\left(\mu^*_{[0,\lambda)}(w,p)\right) = u\left(\mu^*_{\lambda}(w,p)\right)$, indicating that installing $\mu^*_{[0,\lambda)}(w,p)$ or $\mu^*_{\lambda}(w,p)$ results in the same agent's expected utility rate. Therefore the agent is indifferent about installing $\mu^*(w,p) = 0$ or $\mu^*(w,p) = \sqrt{(1+\eta)p_4\lambda} - \lambda$ with expected utility rate $u\left(\mu^*(w,p); w, p\right) = w - w_4$. However the principal would not propose a contract in this case because none of these capacities leads to admissible solutions (see Definition 2.3). For proof see Proposition 4.24. According to Lemma 4.4, $p_4 > 4p_1 \Rightarrow w_4 = 2\sqrt{(1+\eta)p_4\lambda} - \lambda > 0$.

Subsubcase $w \in (0, w_4)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

Subsubcase $w \geq w_4$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

Subcase $p \in (p_4, p_2]$: From Lemma 4.5 part (b), $u\left(\mu_\lambda^*(w,p)\right) > u\left(\mu_{[0,\lambda)}^*(w,p)\right)$, thus the agent's optimal service capacity is $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and $u(\mu^*(w,p)) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to Lemma 4.4, $p > p_4 > 4p_1 \Rightarrow 2\sqrt{(1+\eta)p\lambda} - \lambda > 0$, therefore we further examine the following subcases.

Subsubcase $w \in \left(0, 2\sqrt{(1+\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

Subsubcase $w \geq 2\sqrt{(1+\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

Case $p \in (p_2, 4p_2]$: According to Table 4.1, there is a service capacity that maximizes $u(\mu)$ for $\mu \in (0, \lambda]$ and a service capacity that maximizes $u(\mu)$ for $\mu > \lambda$. Denote the optimal service capacity in $(0, \lambda]$ by $\mu_{(0,\lambda]}^*(w,p)$. From the first order condition $\mu_{(0,\lambda]}^*(w,p) = \sqrt{(1-\eta)p\lambda} - \lambda$ and from (4.3) $u\left(\mu_{(0,\lambda]}^*(w,p)\right) = w - \eta p - 2\sqrt{(1-\eta)p\lambda} + \lambda$. Denote the optimal service capacity for $\mu > \lambda$ by $\mu_\lambda^*(w,p)$. From the first order condition $\mu_\lambda^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and from (4.3) $u\left(\mu_\lambda^*(w,p)\right) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. The agent has to decide which of the two service capacities he installs and he chooses the one with higher expected utility rate. Note that $u\left(\mu_\lambda^*(w,p)\right) - u\left(\mu_{(0,\lambda]}^*(w,p)\right) = \eta p - 2\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)\sqrt{p\lambda}$. According to Lemma 4.8 part (b) and (c), $p > p_2 \geq p_3$, and according to Lemma 4.2 part (a) $u\left(\mu_\lambda^*(w,p)\right) > u\left(\mu_{(0,\lambda]}^*(w,p)\right)$, therefore the agent's optimal service capacity is $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and $u(\mu^*(w,p)) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to Lemma 4.4, $p > p_2 > 4p_1 \Rightarrow 2\sqrt{(1+\eta)p\lambda} - \lambda > 0$, therefore we examine the following subcases.

Subcase $w \in \left(0, 2\sqrt{(1+\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, thus the agent rejects the contract.

Subcase $w \geq 2\sqrt{(1+\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

Case $p > 4p_2$: According to Table 4.1, the service capacity that maximizes $u(\mu)$ must satisfy $\mu > \lambda$. From the first order condition $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and $u(\mu^*(w,p)) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to Lemma 4.4, $p > 4p_2 > 4p_1 \Rightarrow 2\sqrt{(1+\eta)p\lambda} - \lambda > 0$, therefore we examine the following subcases.

Subcase $w \in \left(0, 2\sqrt{(1+\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, thus the agent rejects the contract.

Subcase $w \geq 2\sqrt{(1+\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

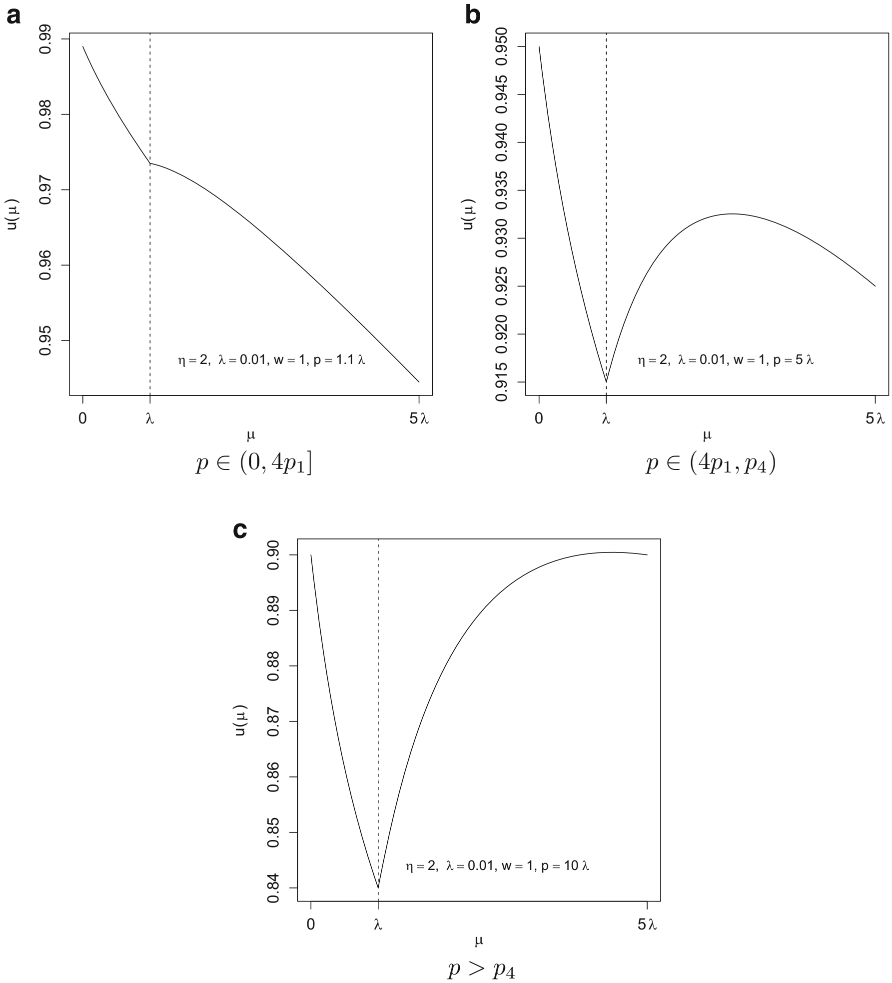


Fig. 4.12 Illustration of the forms of $u(\mu)$ when $\eta \geq 1$

This completes the proof for Proposition 4.23 when $\eta \in [4/5, 1)$.
Case $\eta \geq 1$: According to Lemma 4.22, $p_4 > 4p_1$. Figure 4.12 depicts the shape of $u(\mu)$ when $\eta \geq 1$ and the value of p falls in different ranges. The proof when $\eta \geq 1$ is depicted in Fig. 4.13.

Case $p \in (0, 4p_1]$: According to Table 4.1, $u(\mu)$ is decreasing with respect to $\mu \geq 0$. Thus the agent's optimal service capacity is $\mu^*(w,p) = 0$ and from (4.3) $u(\mu^*(w,p)) = w - p$.

Subcase $w \in (0, p)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

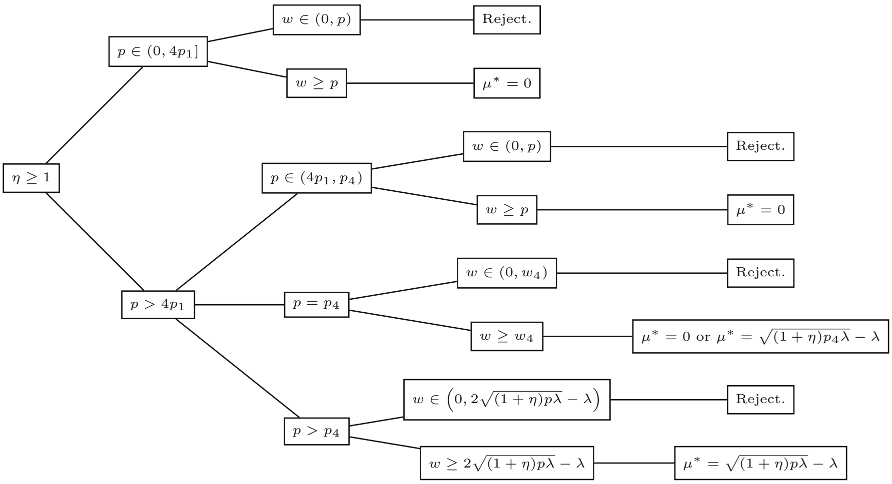


Fig. 4.13 Structure of the proof for Proposition 4.23 when $\eta \geq 1$

Subcase $w \geq p$: $u(\mu^*(w,p)) \geq 0$, thus the agent would accept the contract if offered.

Case $p > 4p_1$: According to Table 4.1, there is a service capacity that maximizes $u(\mu)$ for $\mu \in [0,\lambda)$ and a service capacity that maximizes $u(\mu)$ for $\mu > \lambda$. Denote the optimal service capacity in $[0,\lambda)$ by $\mu^*_{[0,\lambda)}(w,p)$. Note that $u(\mu)$ is decreasing with respect to μ over $[0,\lambda)$, therefore the agent's optimal service capacity is $\mu^*_{[0,\lambda)}(w,p) = 0$ and from (4.3) $u\left(\mu^*_{[0,\lambda)}(w,p)\right) = w - p$. Denote the optimal service capacity for $\mu > \lambda$ by $\mu^*_\lambda(w,p)$. From the first order condition $\mu^*_\lambda(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and from (4.3) $u\left(\mu^*_\lambda(w,p)\right) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. The agent has to decide which of the two service capacities he is going to install and he chooses the one that generates a higher expected utility rate. Note that $u\left(\mu^*_\lambda(w,p)\right) - u\left(\mu^*_{[0,\lambda)}(w,p)\right) = p - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to Lemma 4.22, $p_4 > 4p_1$ and we need to examine the following subcases.

Subcase $p \in (4p_1, p_4)$: According to Lemma 4.6, $4p_1 > \left(1 + 2\eta - 2\sqrt{\eta(1+\eta)}\right)\lambda$ and according to Lemma 4.5 part (a), $u\left(\mu^*_{[0,\lambda)}(w,p)\right) > u\left(\mu^*_\lambda(w,p)\right)$, therefore the agent's optimal service capacity is $\mu^*(w,p) = 0$ and $u(\mu^*(w,p)) = w - p$.

Subsubcase $w \in (0,p)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

Subsubcase $w \geq p$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

Subcase $p = p_4$: According to Lemma 4.5 part (c), $u\left(\mu^*_{[0,\lambda)}(w,p)\right) = u\left(\mu^*_{\lambda}(w,p)\right)$, indicating that installing $\mu^*_{[0,\lambda)}(w,p)$ or $\mu^*_{\lambda}(w,p)$ leads to the same agent's expected utility rate. Therefore the agent is indifferent about installing $\mu^*(w,p) = 0$ or $\mu^*(w,p) = \sqrt{(1+\eta)p_4\lambda} - \lambda$ and in such case $u\left(\mu^*(w,p); w,p\right) = w - w_4$. However the principal would not propose a contract in this case, because none of these capacities leads to admissible solutions (see Definition 2.3). For proof see Proposition 4.24. According to Lemma 4.4, $p_4 > 4p_1 \Rightarrow w_4 = 2\sqrt{(1+\eta)p_4\lambda} - \lambda > 0$.

Subsubcase $w \in (0, w_4)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.

Subsubcase $w \geq w_4$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

Subcase $p > p_4$: From Lemma 4.5 part (b), $u\left(\mu^*_{\lambda}(w,p)\right) > u\left(\mu^*_{[0,\lambda)}(w,p)\right)$, thus the agent's optimal capacity is $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ and $u(\mu^*(w,p)) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$. According to Lemma 4.4, $p > p_4 > 4p_1 \Rightarrow 2\sqrt{(1+\eta)p\lambda} - \lambda > 0$, therefore we further examine the following subcases.

Subsubcase $w \in \left(0, 2\sqrt{(1+\eta)p\lambda} - \lambda\right)$: $u(\mu^*(w,p)) < 0$, thus the agent rejects the contract.

Subsubcase $w \geq 2\sqrt{(1+\eta)p\lambda} - \lambda$: $u(\mu^*(w,p)) \geq 0$, therefore the agent would accept the contract if offered.

□

In summary, given exogenous market conditions such that a mutually beneficial contract with a strongly risk-averse agent exists (see Theorem 4.27 later), only one formula is needed for the agent to compute his optimal service capacity: $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda > 0$.

The conditions when a strongly risk-averse agent accepts the contract can be depicted by the shaded areas in Fig. 4.14, where $\eta = 2$. The two shaded areas with different grey scales represent conditions (4.24) and (4.26) under which the agent accepts the contract but responds differently. The lower bound function of the shaded areas (denoted by $w_0(p)$) represents the set of offers that give the agent zero expected utility rate. $w_0(p)$ is defined as follows:

$$w_0(p) = \begin{cases} p \text{ when } p \in (0, p_4] \\ 2\sqrt{(1+\eta)p\lambda} - \lambda \text{ when } p > p_4 \end{cases}$$

Since $\lim_{p \to p_4^-} w_0(p) = \lim_{p \to p_4^+} w_0(p) = p_4$, $w_0(p)$ is continuous everywhere over interval $p \in \mathbb{R}_+$. However since $\lim_{p \to p_4^-} dw_0(p)/dp = 1 \neq \sqrt{1+\eta}\left(\sqrt{1+\eta} + \sqrt{\eta}\right) = \lim_{p \to p_4^+} dw_0(p)/dp$, $w_0(p)$ is not differentiable at $p = p_4$.

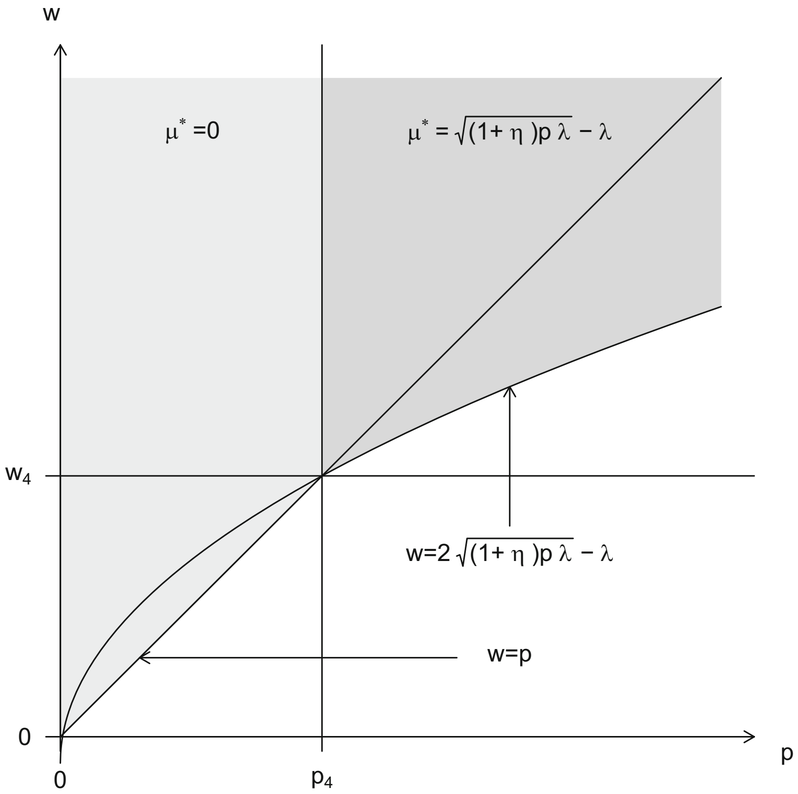


Fig. 4.14 Conditions when a strongly risk-averse agent accepts the contract with $\eta = 2$

4.2.1 Sensitivity Analysis of a Strongly Risk-Averse Agent's Optimal Strategy

Since the principal does not propose a contract that will be responded to with zero service capacity, therefore the only viable case is when the agent in response installs positive service capacity: $\mu^*(w, p) = \sqrt{(1+\eta)p\lambda} - \lambda$. The w is bounded below by $2\sqrt{(1+\eta)p\lambda} - \lambda = \eta p P(1) + pP(1) + \mu^*(w, p)$ (see (4.26)), with $\eta p P(1)$ representing the expected risk rate perceived by the agent and $pP(1)$ representing the expected penalty rate charged by the principal. It indicates that the agent has to be reimbursed for the expected risk rate, the expected penalty rate, and the cost of the optimal service capacity.

The optimal service capacity $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ depends on the penalty rate p, the failure rate λ and the risk coefficient η. Its derivatives are:

$$\frac{\partial \mu^*}{\partial p} = \sqrt{\frac{(1+\eta)\lambda}{4p}} > 0, \ \frac{\partial \mu^*}{\partial \lambda} = \sqrt{\frac{(1+\eta)p}{4\lambda}} - 1 \text{ and } \frac{\partial \mu^*}{\partial \eta} = \sqrt{\frac{p\lambda}{4(1+\eta)}} > 0$$

The derivatives indicate that given λ and η the agent will increase the service capacity when the penalty rate increases. Note that $\sqrt{(1+\eta)p\lambda} - \lambda$, as a function of λ, increases when $(1+\eta)p/4 > \lambda$. From conditions (4.25) and (4.26) the agent installs service capacity $\sqrt{(1+\eta)p\lambda} - \lambda$ when $p \geq p_4$ and from Lemma 4.22 we have $p_4 > 4p_1$. Therefore we have $p > 4p_1 = 4\lambda/(1+\eta) \Rightarrow (1+\eta)p/4 > \lambda \Rightarrow \partial\mu^*/\partial\lambda > 0$. Thus, given p and η, the agent will increase the service capacity when the failure rate increases. Given the penalty rate and the failure rate, the agent will increase the service capacity as his risk-aversion increases.

The agent's optimal expected utility rate when installing capacity $\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda$ is $u_A^* \equiv u_A(\mu^*(w,p); w,p) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$, and it depends on w, p, η and λ. Note that $\partial u_A^*/\partial w = -1 < 0$, $\partial u_A^*/\partial p = -\sqrt{(1+\eta)\lambda/p} < 0$ and $\partial u_A^*/\partial \eta = -\sqrt{p\lambda/(1+\eta)} < 0$, indicating that the agent's optimal expected utility rate decreases with the compensation rate, the penalty rate and his risk intensity. Note that $\partial u_A^*/\partial \lambda = -\left(\sqrt{p} - \sqrt{p_1}\right)/\sqrt{p_1}$, and from Proposition 4.23 $p \geq p_4 > 4p_1 \Rightarrow \sqrt{p} - \sqrt{p_1} > 0$, therefore the agent's optimal expected utility rate also decreases with the failure rate.

Summary: Recall that a risk-neutral agent would accept a contract, install $\mu^*(w,p) = 0$ and receive $u(\mu^*(w,p); w,p) = w - p$ given the set of offers $\{(w,p) : p \in (0,\lambda], w \geq p\}$. Given the set of offers $\left\{(w,p) : p > \lambda, w \geq 2\sqrt{p\lambda} - \lambda\right\}$ he would accept the contract, install $\mu^*(w,p) = \sqrt{p\lambda} - \lambda$ and receive expected utility rate $u(\mu^*(w,p); w,p) = w - 2\sqrt{p\lambda} + \lambda$. By comparing the optimal solutions of a strongly risk-averse agent with that of a risk-neutral agent, two conclusions are drawn:

1. Given a λ, the principal must set a higher p in order to induce a strongly risk-averse agent to install a positive service capacity versus a risk-neutral agent ($p > \lambda$ for risk-neutral agent, $p > \left(1 + 2\eta + 2\sqrt{\eta(1+\eta)}\right)\lambda$ for strongly risk-averse agent).
2. With the same w and p, given that the agent accepts the contract and installs a positive service capacity, the expected utility rate of a strongly risk-averse agent decreases with respect to η since

$$u\left(\mu^*(w,p) = \sqrt{(1+\eta)p\lambda} - \lambda\right) = w - 2\sqrt{(1+\eta)p\lambda} + \lambda$$

$$\Rightarrow \frac{\partial u}{\partial \eta} = -\sqrt{\frac{p\lambda}{1+\eta}} < 0$$

Therefore a strongly risk-averse agent is always strictly worse off than a risk-neutral agent.

Compared to a weakly risk-averse agent, a strongly risk-averse agent has fewer options of positive optimal service capacities (he will never install $\mu^*(w,p) = \sqrt{(1-\eta)p\lambda} - \lambda$ when $\eta \in [4/5, 1)$) because the perceived risk rate is high enough such that the only reasonable choice is to invest more in service capacity to compensate for the risk.

4.2.2 Principal's Optimal Strategy

We now derive the principal's optimal strategy while anticipating the agent's optimal response $\mu^*(w,p)$. For that the principal solves the optimization problem:

$$\max_{w>0,p>0} \Pi_P(w,p;\mu^*(w,p)) = \max_{w>0,p>0} \left\{ \frac{r\mu^*(w,p)}{\lambda + \mu^*(w,p)} - w + \frac{p\lambda}{\lambda + \mu^*(w,p)} \right\} \tag{4.27}$$

and recovers the optimizing values: $(w^*, p^*) = \text{argmax}_{w>0,p>0} \Pi_P(w,p;\mu^*(w,p))$.

Before deriving the principal's optimal strategy, we reexamine the case when the principal offers $p = p_4$ and $w \geq w_4$, under which the agent is indifferent regarding two different service capacities which however effect the principal differently. Since any selected solution $((w,p),\mu)$ has to be an admissible solution (see Definition 2.3) we test the solutions' membership in Proposition 4.24.

Proposition 4.24. *Suppose a strongly risk-averse agent. Assume that the principal's possible offers are constrained to set* $\{(w,p) : p = p_4, w \geq w_4\}$.

(a) *If* $r \in (0, p_4]$*, then the principal does not propose a contract.*
(b) *If* $r > p_4$*, the agent installs* $\sqrt{(1+\eta)p_4\lambda} - \lambda$ *if offered a contract.*

Proof. For $w \geq w_4$ we have $\partial\Pi_P(w,p_4;\mu)/\partial\mu = (r-p_4)\lambda/(\lambda+\mu)^2$. Define $\mu_L \equiv 0$ and $\mu_H \equiv \sqrt{(1+\eta)p_4\lambda} - \lambda$ and note that $\mu_H > \mu_L$. If $r \in (0, p_4)$, then $\partial\Pi_P/\partial\mu < 0$, therefore $((w,p_4),\mu_L) \succeq ((w,p_4),\mu_H)$ and the agent would install μ_L if offered a contract since $((w,p_4),\mu_H)$ is not an admissible solution. However in such case the principal's expected profit rate is $\Pi_P(w,p_4;\mu_L) = -w + p_4 \leq 0$, therefore the principal would not propose a contract. If $r = p_4$, then $\partial\Pi_P/\partial\mu = 0$, therefore the agent installs either μ_L or μ_H if offered a contract. However in such case the principal's expected profit rate is $\Pi_P(w,p_4;\mu_L) = \Pi_P(w,p_4;\mu_H) = -w + p_4 \leq 0$, therefore the principal would not propose a contract. If $r > p_4$, then $\partial\Pi_P/\partial\mu > 0$ and $((w,p_4),\mu_H) \succeq ((w,p_4),\mu_L)$. If the principal offers a contract (where the conditions will be discussed in detail in Theorem 4.27 that follows), then by Definition 2.3 only μ_H leads to admissible solutions. □

Notation:

$$r_4 \equiv \left(1 + 2\eta + 2\sqrt{\eta(1+\eta)}\right)(1+2\eta)\lambda = (1+2\eta)p_4 \tag{4.28}$$

r_4 is a function of λ and η however we suppress the parameters (λ, η).

Next we state several technical lemmas (see proofs in the Appendix).

Lemma 4.25. *Consider* $\max_{x \geq \sqrt{p_4}} f(x)$ *where* $f(x) = r + \lambda - \sqrt{p_1}\,((1+2\eta)x + r/x)$ *and denote* $x^* = \operatorname{argmax}_{x \geq \sqrt{p_4}} f(x)$. *The solutions to this optimization problem are*

(a) $x^* = \sqrt{p_4}$ *if* $r \in (0, r_4]$.
(b) $x^* = \sqrt{r/(1+2\eta)}$ *if* $r > r_4$.

Lemma 4.26. *Let* $\eta > 0$ *and* $\lambda > 0$, *then* $(1+2\eta)p_4 > \left(1 + 3\eta + 2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta)$.

The principal's optimal strategy is derived in Theorem 4.27. Recall that Proposition 4.23 describes the agent's optimal response to each pair of compensation rate and penalty rate $(w, p) \in \mathbb{R}^2_+$. Since the principal will not propose a contract that is going to be rejected by a strongly risk-averse (SRA) agent, therefore Theorem 4.27 only considers pairs $(w, p) \in \mathbb{R}^2_+$ such that the agent receives a non-negative expected utility rate. Define

$$\begin{aligned} \mathfrak{D}_{(4.24)} &\equiv \{(w,p) \text{ that satisfies } (4.24) \text{ when } \eta \geq 4/5\} \\ \mathfrak{D}_{(4.25)} &\equiv \{(w,p) \text{ that satisfies } (4.25) \text{ when } \eta \geq 4/5\} \\ \mathfrak{D}_{(4.26)} &\equiv \{(w,p) \text{ that satisfies } (4.26) \text{ when } \eta \geq 4/5\} \\ \mathfrak{D}_{\text{SRA}} &\equiv \mathfrak{D}_{(4.24)} \cup \mathfrak{D}_{(4.25)} \cup \mathfrak{D}_{(4.26)} \end{aligned} \tag{4.29}$$

Theorem 4.27. *Given a strongly risk-averse agent and* $(w, p) \in \mathfrak{D}_{\text{SRA}}$.

(a) *If* $r \in (0, p_4]$, *then the principal does not propose a contract.*
(b) *If* $r \in (p_4, r_4]$, *then the principal's offer and the capacity installed by the agent are*

$$(w^*, p^*) = (w_4, p_4) \text{ and } \mu^*(w^*, p^*) = \sqrt{(1+\eta)p_4\lambda} - \lambda \tag{4.30}$$

and the principal's expected profit rate is

$$\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p_4} + \frac{r}{\sqrt{p_4}}\right) \tag{4.31}$$

(c) *If $r > r_4$, then the principal's offer and the capacity installed by the agent are*

$$(w^*, p^*) = \left(2\sqrt{\frac{(1+\eta)r\lambda}{1+2\eta}} - \lambda, \frac{r}{1+2\eta}\right) \text{ and } \mu^*(w^*, p^*) = \sqrt{\frac{(1+\eta)r\lambda}{1+2\eta}} - \lambda \tag{4.32}$$

and the principal's expected profit rate is

$$\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r - 2\sqrt{\frac{(1+2\eta)r\lambda}{1+\eta}} + \lambda \tag{4.33}$$

Proof. The structure of the proof is depicted in Fig. 4.15.

Case $(w, p) \in \mathfrak{D}_{(4.24)} \cup \mathfrak{D}_{(4.25)}$: According to Proposition 4.23 part (a), if $(w, p) \in \mathfrak{D}_{(4.24)}$, then in case the principal makes an offer, the agent accepts the contract but does not install any service capacity. Since $\partial \Pi_P / \partial w = -1 < 0$, therefore $w^* = p$ and from Eq. (3.3) $\Pi_P(w^*, p; \mu^*(w^*, p)) = -w^* + p = -p + p = 0$. According to Propositions 4.23 part (b) and 4.24, if $(w, p) \in \mathfrak{D}_{(4.25)}$ (which implies $p = p_4$), then the principal does not propose a contract if $r \in (0, p_4]$, or installs $\sqrt{(1+\eta)p_4\lambda} - \lambda$ in case the principal makes an offer when $r > p_4$. Since $\partial \Pi_P / \partial w = -1 < 0$, therefore $w^* = w_4$. From Proposition 4.23 part (b), if the principal offers a contract with $(w, p) = (w_4, p_4)$, then the agent installs either $\mu^*(w_4, p_4) = 0$ or $\mu^*(w_4, p_4) = \sqrt{(1+\eta)p_4\lambda} - \lambda$. Denote the principal's expected profit rate when $(w, p) = (w_4, p_4)$ and $\mu^*(w, p) = 0$ by $\Pi_P^L(p_4)$, and denote the principal's expected profit rate when $(w, p) = (w_4, p_4)$ and $\mu^*(w, p) = \sqrt{(1+\eta)p_4\lambda} - \lambda$ by $\Pi_P^H(p_4)$. By plugging the value of w, p and μ into Eq. (3.3):

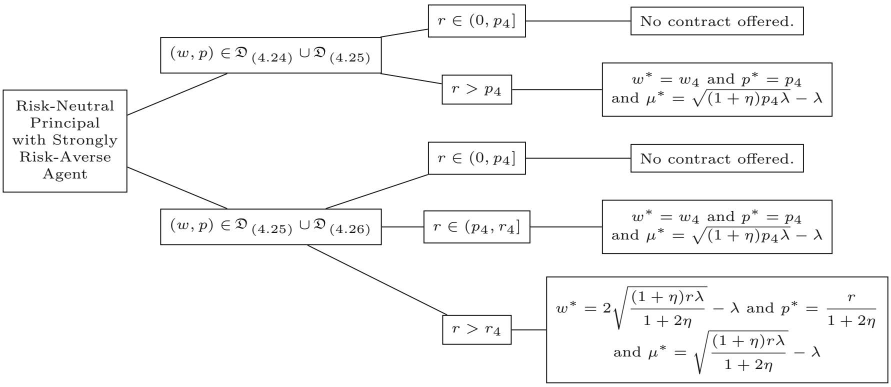


Fig. 4.15 Structure of the proof for Theorem 4.27

$$\Pi_P^L(p_4) = -w_4 + p_4 = 0 \tag{4.34}$$

$$\Pi_P^H(p_4) = r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p_4} + \frac{r}{\sqrt{p_4}}\right) = \left(\frac{\sqrt{p_4} - \sqrt{p_1}}{\sqrt{p_4}}\right)(r - p_4) \tag{4.35}$$

In such case the principal's optimization problem is $\max_{p\in(0,p_4]} \Pi_P(w^*, p; \mu^*(w^*, p))$ where:

$$\Pi_P(w^*, p; \mu^*(w^*, p)) = \begin{cases} 0 \text{ for } p \in (0, p_4) \\ \max\left\{\Pi_P^L(p_4), \Pi_P^H(p_4)\right\} \text{ for } p = p_4 \end{cases}$$

Subcase $r \in (0, p_4]$: By Proposition 4.24 part (a), the principal does not offer a contract.

Subcase $r > p_4$: According to Lemma 4.25 part (b), $p^* = p_4$ and according to Proposition 4.24 part (b) the principal's expected profit rate $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = \Pi_P^H(p_4) > \Pi_P^L(p_4) = 0$. Thus the principal proposes a contract with $w^* = w_4$ and $p^* = p_4$ that induces the agent to install $\mu^*(w^*, p^*) = \sqrt{(1+\eta)p_4\lambda} - \lambda$.

Case $(w, p) \in \mathfrak{D}_{(4.25)} \cup \mathfrak{D}_{(4.26)}$: According to Proposition 4.23 part (c), if $(w, p) \in \mathfrak{D}_{(4.26)}$, then in case the principal makes an offer, the agent accepts the contract and installs $\sqrt{(1+\eta)p\lambda} - \lambda$. Since $\partial\Pi_P/\partial w = -1 < 0$, therefore $w^* = 2\sqrt{(1+\eta)p\lambda} - \lambda$. According to Propositions 4.23 part (b) and 4.24, if $(w, p) \in \mathfrak{D}_{(4.25)}$ (which implies $p = p_4$), then the principal does not propose a contract if $r \in (0, p_4]$, or installs $\sqrt{(1+\eta)p_4\lambda} - \lambda$ in case the principal makes an offer when $r > p_4$. Since $\partial\Pi_P/\partial w = -1 < 0$, therefore $w^* = w_4$. From Proposition 4.23 part (b), if the principal offers a contract with $(w, p) = (w_4, p_4)$, then the agent installs either $\mu^*(w_4, p_4) = 0$ or $\mu^*(w_4, p_4) = \sqrt{(1+\eta)p_4\lambda} - \lambda$. The principal's optimization problem is $\max_{p\geq p_4} \Pi_P(w^*, p; \mu^*(w^*, p))$ where:

$$\Pi_P(w^*, p; \mu^*(w^*, p)) = \begin{cases} \max\left\{\Pi_P^L(p_4), \Pi_P^H(p_4)\right\}, \text{ for } p = p_4 \\ r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p} + \dfrac{r}{\sqrt{p}}\right), \text{ for } p > p_4 \end{cases}$$

Define $x \equiv \sqrt{p}$, the expression $r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p} + r/\sqrt{p}\right)$ can be restated as $f(x) = r + \lambda - \sqrt{p_1}\,((1+2\eta)x + r/x)$. Maximizing $f(x)$ with respect for $x \geq \sqrt{p_4}$ is equivalent to maximizing $r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p} + r/\sqrt{p}\right)$ for $p \geq p_4$ in the sense that

$$\underset{p\geq p_4}{\operatorname{argmax}}\left\{r + \lambda - \sqrt{p_1}\left((1+2\eta)\sqrt{p} + \frac{r}{\sqrt{p}}\right)\right\} = \left(\underset{x\geq\sqrt{p_4}}{\operatorname{argmax}} f(x)\right)^2$$

Since $r_4 = (1+2\eta)p_4 > p_4$, we examine the following subcases.

Subcase $r \in (0, p_4]$: By Proposition 4.24 part (a), the principal does not propose a contract.

Subcase $r \in (p_4, r_4]$: According to Lemma 4.25 part (a), $p^* = p_4$. According to Proposition 4.24 part (b), $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = \Pi_P^H(p_4) > \Pi_P^L(p_4) = 0$. Therefore the principal proposes a contract with $w^* = w_4$ and $p^* = p_4$ that induces the agent to install $\mu^*(w^*, p^*) = \sqrt{(1+\eta)p_4\lambda} - \lambda$.

Subcase $r > r_4$: According to Lemma 4.25 part (b), $p^* = r/(1+2\eta)$ and the principal's expected profit rate is $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r - 2\sqrt{(1+2\eta)r\lambda/(1+\eta)} + \lambda$. According to Lemmas 4.14 and 4.26 $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) > 0$, therefore the principal proposes a contract with $w^* = 2\sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$ and $p^* = r/(1+2\eta)$ that induces the agent to install service capacity $\mu^*(w^*, p^*) = \sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$.

To summarize, if $r \in (0, p_4]$, then the principal does not propose a contract. This case corresponds to Theorem 4.27 (a). If $r \in (p_4, r_4]$, then the principal offers $(w^*, p^*) = (w_4, p_4)$ and the agent installs capacity $\mu^*(w^*, p^*) = \sqrt{(1+\eta)p_4\lambda} - \lambda$. This case corresponds to Theorem 4.27 (b). Finally if $r > r_4$, then according to Lemma 4.25 part (b), the principal offers $(w^*, p^*) = \left(2\sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda, r/(1+2\eta)\right)$ and the agent installs capacity $\mu^*(w^*, p^*) = \sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$. This case corresponds to Theorem 4.27 (c). □

Theorem 4.27 indicates that the existence of a beneficial contract for strongly risk-averse agent is determined exogenously by the market (the revenue rate r), the nature of the equipment (the failure rate λ) and the nature of the agent (the risk coefficient η).

4.3 Risk-Averse Agent: A Summary

Recall the definition of p_2, p_3, p_4, r_2, r_3 and r_4 from (4.5), (4.12), (4.23) and (4.28). The conditions that a principal makes offers to a risk-averse agent is depicted by the shaded areas in Fig. 4.16. The horizontal axis represents the agent's risk coefficient, and the vertical axis represents the revenue rate generated by the principal's unit, which is exogenously determined by the market. The principal makes different offers to the agent when (r, η) is in the five shaded areas with different gray scales. We define

$$p_{34} \equiv \begin{cases} p_3 \text{ for } \eta \in (0, 4/5) \\ p_4 \text{ for } \eta \geq 4/5 \end{cases} \quad \text{and } r_{34} \equiv \begin{cases} r_3 \text{ for } \eta \in (0, 4/5) \\ r_4 \text{ for } \eta \geq 4/5 \end{cases}$$

Note that $\lim_{\eta \to (4/5)^-} r_{34} = 13\lambda = \lim_{\eta \to (4/5)^+} r_{34}$, and note that $\lim_{\eta \to (4/5)^-} \partial r_{34}/\partial\eta = 125\lambda/6 = \lim_{\eta \to (4/5)^+} \partial r_{34}/\partial\eta$, therefore r_{34} is continuous and differentiable everywhere over $\mathbb{R}_+$. Since $\lim_{\eta \to (4/5)^-} p_{34} = 5\lambda = \lim_{\eta \to (4/5)^+} p_{34}$ and $\lim_{\eta \to (4/5)^-} \partial p_{34}/\partial\eta = 25\lambda/6 = \lim_{\eta \to (4/5)^+} \partial p_{34}/\partial\eta$, therefore p_{34} is

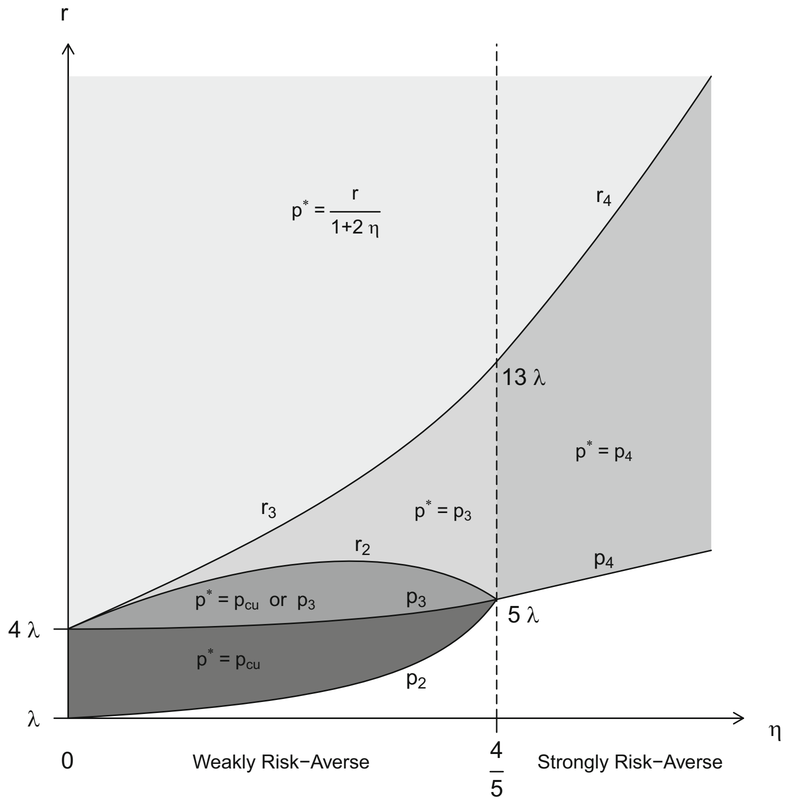


Fig. 4.16 Conditions when a risk-neutral principal makes offers to a risk-averse agent

continuous and differentiable everywhere over $\mathbb{R}_+$ as well. Furthermore, note that $\lim_{\eta\to 0^+} p_3 = \lim_{\eta\to 0^+} r_2 = \lim_{\eta\to 0^+} r_3 = 4\lambda$ and $\lim_{\eta\to(4/5)^-} r_2 = \lim_{\eta\to(4/5)^-} p_2 = 5\lambda$.

4.3.1 Sensitivity Analysis of Optimal Strategies in High Revenue Industry

The revenue rate r is determined exogenously by the market, and consider $r > r_{34}$ (high revenue rate). Equations (4.21) and (4.32) are the second-best solutions when the agent is weakly and strongly risk-averse respectively, and they have the same functional form.

The risk-averse agent's optimal strategy is examined first. Note that the optimal service capacity of a risk-averse agent ($\mu^* = \sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$) is a function of r, λ, and η. The derivatives of μ^* with respect to the parameters are $\partial\mu^*/\partial r = \sqrt{(1+\eta)\lambda}/\left(2\sqrt{(1+2\eta)r}\right) > 0$, $\partial\mu^*/\partial\lambda = \sqrt{(1+\eta)r}/\left(2\sqrt{(1+2\eta)\lambda}\right)-1$ and $\partial\mu^*/\partial\eta = -\sqrt{r\lambda}/\left(2\sqrt{(1+\eta)(1+2\eta)^3}\right) < 0$. The derivatives indicate that given λ and η, the optimal capacity increases when the revenue rate increases, and therefore the average downtime of the principal's unit decreases. Given the revenue rate and the failure rate, the average downtime of the principal's equipment will increase as the agent becomes more risk-averse. Note that $\mu^* = \sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$, as a function of λ, increases when $(1+\eta)r/4(1+2\eta) > \lambda$. According to Lemma 4.3 we have $p_3 > 4p_1$ and according to Lemma 4.22 we have $p_4 > 4p_1$. Furthermore, since we assume that $r > r_{34}$, then $r > r_3 = (1+2\eta)p_3 \Rightarrow r/(1+2\eta) > p_3 \Rightarrow r/(1+2\eta) > 4p_1 = 4\lambda/(1+\eta) \Rightarrow (1+\eta)r/4(1+2\eta) > \lambda$ if $\eta \in (0, 4/5)$ and $r > r_4 = (1+2\eta)p_4 \Rightarrow r/(1+2\eta) > p_4 \Rightarrow r/(1+2\eta) > 4p_1 = 4\lambda/(1+\eta) \Rightarrow (1+\eta)r/4(1+2\eta) > \lambda$ if $\eta \geq 4/5$. Thus, given the revenue rate and the risk coefficient, the failure rate is low compared to the revenue rate, and the average downtime of the principal's equipment will decrease when the failure rate increases.

Next we examine the principal's optimal strategy. Note that the optimal compensation rate of a principal with a risk-averse agent ($w^* = 2\sqrt{(1+\eta)r\lambda/(1+2\eta)}-\lambda$) is a function of r, λ, and η. The derivatives of w^* with respect to the parameters are $\partial w^*/\partial r = \sqrt{(1+\eta)\lambda/(1+2\eta)r} > 0$, $\partial w^*/\partial\lambda = \sqrt{(1+\eta)r/(1+2\eta)\lambda} - 1$ and finally $\partial w^*/\partial\eta = -\sqrt{r\lambda/(1+\eta)(1+2\eta)^3} < 0$. The derivatives indicate that given the λ and η, the optimal compensation rate increases with respect to r. Given the r and the λ, the optimal compensation rate decreases as the agent becomes more risk-averse. Note that $w^* = 2\sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$, as a function of λ, increases when $(1+\eta)r/(1+2\eta) > \lambda$. According to Lemma 4.3 we have $p_3 > 4p_1 > p_1$ and according to Lemma 4.22 we have $p_4 > 4p_1 > p_1$. Furthermore, since we assume that $r > r_{34}$, then $r > r_3 = (1+2\eta)p_3 \Rightarrow r/(1+2\eta) > p_3 \Rightarrow r/(1+2\eta) > p_1 = \lambda/(1+\eta) \Rightarrow (1+\eta)r/(1+2\eta) > \lambda$ if $\eta \in (0, 4/5)$ and $r > r_4 = (1+2\eta)p_4 \Rightarrow r/(1+2\eta) > p_4 \Rightarrow r/(1+2\eta) > p_1 = \lambda/(1+\eta) \Rightarrow (1+\eta)r/(1+2\eta) > \lambda$ if $\eta \geq 4/5$. Therefore the failure rate is low compared to the revenue rate ($(1+\eta)r/(1+2\eta) > \lambda \Rightarrow \partial w^*/\partial\lambda > 0$), indicating that the w^* increases with respect to the failure rate.

The principal's optimal p^* given a risk-averse agent ($p^* = r/(1+2\eta)$) is a function of r and η. Note that p^* is independent of the failure rate λ under the assumption that the revenue rate is sufficiently high compared to the failure rate. The derivatives of p^* with respect to the parameters are $\partial p^*/\partial r = 1/(1+2\eta) > 0$ and $\partial p^*/\partial\eta = -2r/(1+2\eta)^2 < 0$. The derivatives indicate that given the risk η, the optimal penalty p^* increases with respect to r, and given r, the p^* decreases with respect to η.

The principal's optimal expected profit rate given a risk-averse agent

$$\Pi_P^* \equiv \Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r - 2\sqrt{(1+2\eta)r\lambda/(1+\eta)} + \lambda$$

is a function of r, λ, and η. The derivatives of Π_P^* with respect to these parameters are $\partial\Pi_P^*/\partial r = 1 - \sqrt{(1+2\eta)\lambda/(1+\eta)r}$, $\partial\Pi_P^*/\partial\lambda = 1 - \sqrt{(1+2\eta)r/(1+\eta)\lambda}$, $\partial\Pi_P^*/\partial\eta = -\sqrt{r\lambda/(1+2\eta)(1+\eta)^3} < 0$. The derivatives indicate that given r and λ, the principal's optimal expected profit rate decreases as the agent becomes more risk-averse. Note that $\Pi_P^* = r - 2\sqrt{(1+2\eta)r\lambda/(1+\eta)} + \lambda$, as a function of λ, decreases when $(1+2\eta)r/(1+\eta) > \lambda$, and as a function of r, increases when $r > (1+2\eta)\lambda/(1+\eta)$. According to Lemma 4.3 we have $p_3 > 4p_1 > p_1$ and according to Lemma 4.22 we have $p_4 > 4p_1 > p_1$. Furthermore, since we assume that $r > r_{34}$, then $r > r_3 = (1+2\eta)p_3 \Rightarrow r/(1+2\eta) > p_3 \Rightarrow r/(1+2\eta) > p_1 = \lambda/(1+\eta)$ if $\eta \in (0, 4/5)$ and $r > r_4 = (1+2\eta)p_4 \Rightarrow r/(1+2\eta) > p_4 \Rightarrow r/(1+2\eta) > p_1 = \lambda/(1+\eta)$ if $\eta \geq 4/5$. Therefore given a λ and an η, the revenue rate is high compared to the failure rate ($r > (1+2\eta)\lambda/(1+\eta) \Rightarrow \partial\Pi_P^*/\partial r > 0$), thus the principal's optimal expected profit rate increases with respect to the revenue rate. Note that since $\eta > 0$, therefore $r > (1+2\eta)\lambda/(1+\eta) > (1+\eta)\lambda/(1+2\eta) \Rightarrow (1+2\eta)r/(1+\eta) > \lambda$, which implies that given an r and η, the failure rate is low compared to the revenue rate ($(1+2\eta)r/(1+\eta) > \lambda \Rightarrow \partial\Pi_P^*/\partial\lambda < 0$), therefore the principal's optimal expected profit rate decreases with respect to λ.

4.3.2 The Second-Best Solution in High Revenue Industry

By comparing the second-best solution given a risk-averse agent ((4.21) and (4.32)) with the second-best given a risk-neutral agent when $r > r_{34}$, four conclusions are drawn.

1. The optimal w^* and the optimal p^* decrease when the agent is risk-averse versus risk-neutral agent ($w^* : 2\sqrt{r\lambda} - \lambda > 2\sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$ and $p^* : r > r/(1+2\eta)$). It indicates that the risk adds an incentive for the agent to install a higher service capacity by coupling it to the penalty charge collected by the principal.
2. The principal is worse off with a risk-averse agent than a risk-neutral agent ($r - 2\sqrt{r\lambda} + \lambda > r - 2\sqrt{(1+2\eta)r\lambda/(1+\eta)} + \lambda$), as well as with an agent whose action is contractible (recall that the principal receives the same expected profit rate with a risk-neutral agent in first-best and second-best setting). This conclusion is consistent with Proposition 3 part (ii) in Harris and Raviv (1978). The principal's loss can be explained as follows: On one hand, the decrease in the agent's optimal capacity when risk-averse reduces the revenue performance of the principal's unit. At the same time, the monetary equivalency of the risk perceived by the agent is not channeled to the principal, although from the agent's perspective it serves as part of the penalty charge.
3. The μ^* of a risk-averse agent is strictly less than that of a risk-neutral agent ($\sqrt{r\lambda} - \lambda > \sqrt{(1+\eta)r\lambda/(1+2\eta)} - \lambda$). Recall that when the agent is risk-neutral, the μ^* in the second-best solution is the same as that in the first-best solution, indicating that the unobservability of the agent's service capacity does

not contribute to the decrease of the optimal service capacity. When the agent is risk-averse, he compensates for the risk he bears by reducing μ.

4. Given the compensation rate and penalty rate, both weakly and strongly risk-averse agents are worse off compared to a risk-neutral agent.

To summarize, for a principal with high revenue generating unit, agent's risk-aversion reduces the efficiency of the contract (compared to the first-best contract), and therefore it reduces the social welfare.